மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

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தொலைநிலை தொடர் கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

B.Sc. MATHEMATICS

II YEAR

COMPUTATIONAL MATHEMATICS

Sub. Code: JSMA31

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B.Sc. MATHEMATICS – II YEAR
JSMA31: COMPUTATIONAL MATHEMATICS

SYLLABUS

UNIT-I:

ALGEBRAIC AND TRANSCENDENTAL EQUATIONS: Errors in Numerical Computation – Iteration Method – Regula Falsie method.

UNIT-II:

ALGEBRAIC AND TRANSCENDENTAL EQUTIONS: Bisection method – Newton Raphson method – Horner's method.

UNIT-III:

SIMULTANEOUS EQUATIONS: Back substitution – Gauss Elimination method – Gauss Jordan Elimination method – Calculation of inverse of a matrix.

UNIT-IV:

SIMULTANEOUS EQUATIONS: Iterative Methods – Gauss Jacobi iteration method – Gauss-seidel Iteration Method – Relaxation method – Newton Raphson method for simultaneous equations.

UNIT-V:

NUMERICAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS: Classification of partial differential equations of second order – Finite Differnce Approximations to Derivatives – Laplace equation – Poisson's equation.

Recommended Text:

S. Arumugan, A. Thangpandi Isaac and A. Somasundaram, ***Numerical Methods***, Scitech, 2017.

JASMA31: COMPUTATIONAL MATHEMATICS

SYLLABUS

Algebraic and Transcendental Equations

Unit I

1. Introduction

If $f(x)$ is a polynomial then the equation $f(x)$ is called and **algebraic equation.**

Equations which involve transcendental functions like sin x, cos x, tan x, log x, e^x etc. are called **transcendental** equations.

$x^2 + 5x + 6 = 0$; $2x^3 - x + 4 = 0$; $x^5 - x^3 + 3x + 3 = 0$

are some examples of *algebraic equations*.

$2e^x + 1 = 0$; $2x + \cos x - 1 = 0$; $\log_{10} x - 2x = 12$; $a + b \sin x + c \cos x + d \log x = 10$; $x^2 + \log e^{x} - 12 = 0$ are some examples of *transcendental equations*.

If $f(x)$ *is continuous in the interval* $[a, b]$ *and if* $f(a)$ *and* $f(b)$ *are of opposite signs then the equation* $f(x) = 0$ *has at least one root lying between a and b.*

1.1 Errors in Numerical Computation

Definition. If α is an approximate value of a quantity whose exact value is a, then the difference $\epsilon = \alpha - a$ is called the **absolute error** of α or simply the **error** of α.

The **relative error** ϵ_γ is defined by $\epsilon_\gamma = \dfrac{\in}{a} = \dfrac{\alpha - a}{a}$ provided $a \neq 0$. The percentage error ϵ_p is defined by $\epsilon_p = 100\, \epsilon_r$.

$r = a - \alpha = -\epsilon$ is called the **correction**. Thus $a = \alpha + r$.

i.e. True value = approximated value + correction.

Different types of errors in numerical computation.

An **experimental error** is an error present in the given data. Such errors may arise from measurements.

In an iterative computational method the sequence of computational steps necessary to produce an exact result is truncated after a finite number of steps. An error arising out of such truncation is called ***truncation error***.

The process of cutting of digits and retaining required number of digits is called **rounding off**. Errors arising from the process of rounding off during computation are called **round-off errors**.

1.2 Iteration method (Successive approximation method)

Let $f(x) = 0$ be the given equation, algebraic or transcendental. Suppose the equation can be expressed in the form

$$x = \phi(x) \qquad \ldots\ldots (1)$$

where $\phi(x)$ is a continuous function.

Let x_0 be an approximate value of the desired root. We define

$$x_1 = \phi(x_0), \quad x_2 = \phi(x_1), \ldots\ldots, \quad x_n = \phi(x_{n-1})$$

The sequence $(x_0, x_1, \ldots\ldots, x_n, \ldots\ldots)$ is called the sequence of successive approximations.

Suppose the sequence (x_n) converges to α.

$$\text{(i.e.)} \quad \lim_{n\to\infty} x_n = \alpha \qquad \ldots\ldots (2)$$

Since $\phi(x)$ is a continuous function we have $\lim_{n\to\infty} \phi(x_n) = \phi(\alpha)$

$$\lim_{n\to\infty} (x_{n+1}) = \phi(\alpha)$$

$$\therefore \alpha = \phi(\alpha) \text{ (from 2)}$$

Thus α is a root of the equation $x = \phi(x)$ and hence it is a root of $f(x) = 0$.

Order of Convergence

Consider the equation f(x)=0. Suppose the equation can be expressed in the form $x=\varphi(x)$.

Let $x_{n+1}=\varphi(x_n)$ define an iteration method for solving the equation f(x)=0.

Let α be the root of the above equation . Let $x_n = \alpha + \varepsilon_n$.

Then $\varepsilon_n = x_n - \alpha$ is the error in x_n. If $\varphi(x)$ is differentiable any number of times then the Taylor's formula for $\varphi(x)$ isgiven by

$$\Phi(x) = \varphi(\alpha) + \frac{\varphi'(\alpha)}{1!}\varepsilon_n + \frac{\varphi''(\alpha)}{2!}\varepsilon_n^{\,2} + \ldots..$$

The power of ε_n in the first non vanishing term after $\varphi(\alpha)$ is called the order of the iteration process.

Remark : The order is a measure for the speed of convergence.

For example in the case of **first order convergence**

$$x_{n+1} = \phi(\alpha) + \phi'(\alpha)\epsilon_n \text{ (omitting higher powers)}$$

$$\therefore x_{n+1} = \alpha + \phi'(\alpha)\epsilon_n$$

(i.e.) $$x_{n+1} - \alpha = \phi'(\alpha)\varepsilon_n$$

$$\therefore \epsilon_{n+1} = \phi'(\alpha)\varepsilon_n.$$

Similarly in the case of **second order convergence,** $\epsilon_{n+1} = \frac{1}{2}\phi''(\alpha)\varepsilon_n^2$.

Problem 1. Use the method of iteration to find the real root lying between 1 and 2 of the equation $x^3 - 3x + 1 = 0$.

Solution. Let $f(x) = x^3 - 3x + 1$. Here $f(1) = -1$ and $f(2) = 3$.

$\therefore$ One root of $f(x) = 0$ lies between 1 and 2.

Now, $f(x) = 0$ gives $x^3 = 3x - 1$.

$$\therefore x = (3x-1)^{\frac{1}{3}} = \phi(x)$$

$$\therefore \phi'(x) = \frac{1}{(3x-1)^{\frac{2}{3}}}$$

Clearly $|\phi'(x)| < 1$ for all $x \in (1, 2)$. Hence if we take $x_0 = 2$, then the sequence of successive approximations $x_0, x_1, \ldots\ldots, x_n \ldots$ is convergent. Let $x_0 = 2$.

Now $$x_1 = \phi(x_0) = \phi(2) = 5^{\frac{1}{3}} = 1.7100$$

$$x_2 = \phi(x_1) = \phi(1.71) = (3\times1.71-1)^{\frac{1}{3}} = (4.13)^{\frac{1}{3}} = 1.6044$$

$$x_3 = \phi(x_2) = \phi(1.6044) = (3\times1.6044-1)^{\frac{1}{3}} = (3.8132)^{\frac{1}{3}} = 1.5623$$

$$x_4 = \phi(x_3) = \phi(1.5623) = (3\times1.5623-1)^{\frac{1}{3}} = (3.6869)^{\frac{1}{3}} = 1.5449$$

$$x_5 = \phi(x_4) = \phi(1.5449) = (3\times1.5449-1)^{\frac{1}{3}} = (3.6347)^{\frac{1}{3}} = 1.5375$$

$$x_6 = \phi(x_5) = \phi\,(1.5375) = (3\times1.5375-1)^{\frac{1}{3}} = (3.6125)^{\frac{1}{3}} = 1.5344$$

$$x_7 = \phi(x_6) = \phi\,(1.5344) = (3\times1.5344-1)^{\frac{1}{3}} = (3.6032)^{\frac{1}{3}} = 1.5331$$

$$x_8 = \phi(x_7) = \phi\,(1.5331) = (3\times1.5331-1)^{\frac{1}{3}} = (3.5993)^{\frac{1}{3}} = 1.5325$$

$$x_9 = \phi(x_8) = \phi\,(1.5325) = (3\times1.5325-1)^{\frac{1}{3}} = (3.5975)^{\frac{1}{3}} = 1.5323$$

$$x_{10} = \phi(x_9) = \phi\,(1.5323) = (3\times1.5323-1)^{\frac{1}{3}} = (3.5969)^{\frac{1}{3}} = 1.5322.$$

Now $x_9 = x_{10}$, upto 3 places of decimals.

Hence approximate value of the required root is 1.532

Problem 2. Use the method of iteration to solve the equation $3x - \log_{10} x = 6$.

Solution: Let $f(x) = 3x - \log_{10} x - 6$

$$f(2) = 6 - 0.3010 - 6 = -0.3010$$

$$f(3) = 9 - 0.4771 - 6 = -2.5229$$

$\therefore$ One root of $f(x) = 0$ lies between 2 and 3.

Now, $f(x) = 0$ can be written as

$$x = \frac{1}{3}(6 + log_{10} x) = \phi\,(x).$$

Now, $\phi'(x) = \frac{1}{3}\left(\frac{\log_{10} e}{x}\right)$ $\quad$ ($\because \log_{10} x = \log_e x \times \log_{10} e$)

$$|\phi'(x)| = \frac{0.4343}{3}\left|\frac{1}{x}\right| \qquad (\because \log_{10} e = 0.4343)$$

$$< 1 \text{ for all } x \in (2, 3).$$

Hence if we take $x_0 = 2$, then the sequence of successive approximations $x_0, x_1, x_2 \ldots$ is convergent.

Now $x_1 = \phi(x_0) = \frac{1}{3}(6+log_{10}2) = \frac{6.3010}{3} = 2.1003$

$$x_2 = \phi(x_1) = \frac{1}{3}(6+log_{10}2.1003) = \frac{6.3223}{3} = 2.1074$$

$$x_3 = \phi(x_2) = \frac{1}{3}(6+log_{10}2.1076) = \frac{6.3238}{3} = 2.1079$$

$$x_4 = \phi(x_3) = \frac{1}{3}(6+log_{10}2.1079) = \frac{6.3239}{3} = 2.1080$$

$$x_5 = \phi(x_4) = \frac{1}{3}(6+log_{10}2.1080) = \frac{6.3239}{3} = 2.1080$$

Hence approximate value of the required root is 2.108

Problem 3. Find the real root of the equation cos $x = 3x - 1$ correct to four decimal places using successive approximation method.

Solution: Let $f(x) = 3x - 1 - \cos x.$

$$f(0) = 0 - 1 - 1 = -2$$

and $f\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} - 1 - 0 = 4.7143 - 1 = 3.7143$

$\therefore$ One root of $f(x) = 0$ lies between 0 and $\frac{\pi}{2}$.

$f(x) = 0$ can be written as $x = \frac{1}{3}(1+\cos x) = \phi(x)$.

Now, $\phi'(x) = -\frac{\sin(x)}{3}$

$\therefore |\phi'(x)| = \left|\frac{\sin x}{3}\right| < 1$ for all x in $\left(0, \frac{\pi}{2}\right)$.

Hence if we take $x_0 = 0$, then the sequence of successive approximations $x_0, x_1, x_2, \ldots, x_n, \ldots$ is convergent.

Now $x_1 = \phi(x_0) = \phi\,(0) = \frac{1}{3}(1+\cos 0) = 0.6667$

$x_2 = \phi(x_1) = \phi\,(0.6667) = \frac{1}{3}(1+0.7859) = 0.5953$

$x_3 = \phi(x_2) = \phi\,(0.5953) = \frac{1}{3}(1+0.8280) = 0.6093$

$x_4 = \phi(x_3) = \phi\,(0.6093) = \frac{1}{3}(1+0.8200) = 0.6067$

$x_5 = \phi(x_4) = \phi\,(0.6067) = \frac{1}{3}(1+0.8215) = 0.6072$

$x_6 = \phi(x_5) = \phi\,(0.6072) = \frac{1}{3}(1+0.8212) = 0.6071$

$x_7 = \phi(x_6) = \phi\,(0.6071) = \frac{1}{3}(1+0.8214) = 0.6071.$

Hence approximate value of the required root is 0.6071.

Problem 4. Can we find a real root of the equation $x^3 + x^2 - 1 = 0$ in the interval [0, 1] by the method of iteration?

Solution. Writing $x^3 + x^2 - 1 = 0$ as $x^2(x+1) = 1$, we get $x = \frac{1}{\sqrt{x+1}}$.

Let $\phi\,(x) = \frac{1}{\sqrt{x+1}}$. Then $\phi'(x) = -\frac{1}{2(x+1)^{\frac{3}{2}}}$.

Clearly $|\,\phi'(x)\,| = \frac{1}{2(x+1)^{\frac{3}{2}}} < 1$ for all $x \in [0, 1]$.

Hence if we take $x_0 = 0$ (or any number in [0, 1]), then the sequence of approximations $x_0, x_1, x_2, \ldots, x_n, \ldots$ is convergent and an approximate value of the root can be obtained by the method of iteration.

Problem 5. Can we apply iteration method to find the root of the equation $2x = \cos x + 3$ in $\left[0, \frac{\pi}{2}\right]$?

Solution. $2x = \cos x + 3$ gives $x = \frac{1}{2}(\cos x + 3)$.

Let $\phi(x) = \frac{1}{2}(\cos x + 3)$. Then $|\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1$ for all x in $\left[0, \frac{\pi}{2}\right]$.

Hence by taking $x_0 = 0$ (or any point in $\left[0, \frac{\pi}{2}\right]$), the sequence of approximations $x_0, x_1, x_2, \ldots, x_n, \ldots$ is convergent and an approximate value of the root can be obtained by the method of iteration.

Exercises

Solve the following equations using iterative method

1. $x^3 + x^2 - 1 = 0$
2. Find the negative root of $x^3 - 2x + 5 = 0$

1.3 Regula Falsi Method (Method of false position).

Consider the equation $f(x) = 0$ where $f(x)$ is a continuous function. Choose two points a, b such that $f(a)$ and $f(b)$ are of opposite signs. Hence there exists a root lying between a and b. In this method we approximate the curve of the function f(x) by a chord .The point of intersection of the chord with the x – axis is taken as the first approximation x_1 to the root.

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

Now if $f(a)$ and $f(x_1)$ are of opposite signs then the root lies between a and x_1. So we replace b by x_1 in (2) and get the next approximation x_2.

But if $f(a)$ and $f(x_1)$ are of same sign, then $f(x_1)$ and $f(b)$ will be of opposite signs. Hence the root lies between x_1 and b. We replace a by x_1.

The process is repeated until the root is found to the desired accuracy.

Problem 1. Find the real root lying between 1 and 2 of the equation $x^3 - 3x + 1 = 0$ upto 3 places of decimals by using Regula – falsi method.

Solution. Let $f(x) = x^3 - 3x + 1$. Since $f(1) = -1$, $f(2) = 3$ one root lies between 1 and 2.

Let $a = 1$ and $b = 2$. The first approximation is

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1\times 3 - 2(-1)}{3+1} = \frac{5}{4} = 1.25.$$

Now $f(x_1) = f(1.25) = 0.7969$

Since $f(1.25)$ is negative and $f(2)$ is positive the root lies between 1.25 and 2.

Let $a = 1.25$ and $b = 2$.

The second approximation is $x_2 = \dfrac{1.25\times 3 - 2(-0.7969)}{3+0.7969} = 1.4074.$

Now $f(1.4074) = 1.4074^3 - 3 \times 1.4074 + 1 = -0.4345.$

Since $f(1.4074)$ is negative and $f(2)$ is positive the root lies between 1.4074 and 2.

Take $a = 1.4074$ and $b = 2$.

The third approximation is $x_3 = \dfrac{1.4074\times 3 - 2(-0.4345)}{3+0.4345} = \dfrac{4.2222+0.8690}{3.4345} = 1.4824.$

Now $f(1.4824) = 1.4824^3 - 3 \times 1.4824 + 1 = 3.2576 - 4.4472 + 1 = -0.1896.$

Since $f(1.4824)$ is negative and $f(2)$ is positive the root lies between 1.4824 and 2. Let $a = 1.4824$ and $b = 2$.

The fourth approximation is $x_4 = \dfrac{1.4824\times 3 - 2\times(-0.1896)}{3+0.1896} = 1.5132.$

Now $f(1.5132) = 1.5132^3 - 3 \times 1.5132 + 1 = 3.4649 - 4.5396 + 1 = -0.0747.$

Since $f(1.5132)$ is negative and $f(2)$ is positive the root lies between 1.5132 and 2. Let $a = 1.5132$ and $b = 2$.

The fifth approximation is $x_5 = \dfrac{1.5132\times 3 - 2(-0.0747)}{3+0.0747} = 1.525.$

Now $f(1.525) = 1.525^3 - 3 \times 1.525 + 1 = 3.5466 - 4.575 + 1 = -0.0284.$

Since f (1.525) is negative and f (2) is positive the root lies between 1.525 and 2. Take a = 1.525 and b = 2.

The sixth approximation is $x_6 = \frac{1.525 \times 3 - 2 \times (-0.0284)}{3 + 0.0284}$ =1.5295.

Now f (1.5295) = 1.5295^3 – 3 × 1.5295 + 1 = 3.5781 – 4.5885 +1= – 0.0104.

Since f (1.5295) is negative and f (2) is positive the root lies between 1.5295 and 2. Take a = 1.5295 and b = 2.

∴ The seventh approximation is $x_7 = \frac{1.5295 \times 3 - 2 \times (-0.0104)}{3 + 0.0104}$ =1.5311.

Now f (1.5311) = 1.5311^3 – 3 × 1.5311 + 1 = 3.5893 – 4.5993 + 1= – 0.0004.

Since f (1.5311) is negative and f (2) is positive the root lies between 1.5311 and 2. Take a = 1.5311 and b = 2.

∴ The eighth approximation is $x_8 = \frac{1.5311 \times 3 - 2 \times (-0.0040)}{3 + 0.0040}$ =1.5317.

Now f (1.5317) = 1.5317^3 – 3 × 1.5317 + 1 = 3.5935 – 4.5951 + 1= – 0.0016.

Since f (1.5317) is negative and f (2) is positive the root lies between 1.5317 and 2. Take a = 1.5317 and b = 2.

∴ The ninth approximation is $x_9 = \frac{1.5317 \times 3 - 2 \times (-0.0016)}{3 + 0.0016}$ =1.5319.

= 1.532 (corrected upto 3 places of decimals)

$$\therefore\ x_8 = x_9 = 1.532$$

∴ The required root is 1.532 (corrected upto 3 places of decimals)

Problem 2. Find a root of the equation $x^3 - 3x - 5 = 0$ by the method of false position.

Solution. Let $f(x) = x^3 - 3x - 5$. Here $f(2) = -3$, $f(3) = 13$.

∴ The root lies between 2 and 3. Let a = 2 and b = 3.

The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2\times 13 - 3\times(-3)}{13+3} = 2.1875.$$

Now $f(x_1) = f(2.1875) = 10.4675 - 6.5625 - 5 = -1.095$.

Hence the root lies between 2.1875 and 3. Let a = 2.1875 and b = 3.

Hence the second approximation is given by

$$x_2 = \frac{2.1875\times 13 - 3\times(-1.095)}{13+1.095} = 2.2506.$$

Now $f(x_2) = f(2.2506) = 11.3997 - 6.7518 - 5 = -0.3521$.

Hence the root lies between 2.2506 and 3. Let a = 2.2506 and b = 3.

Hence the third approximation is given by

$$x_3 = \frac{2.2506\times 13 - 3\times(-0.3521)}{13+0.3521} = 2.2704.$$

Now $f(x_3) = f(2.2704) = 11.7033 - 6.8112 - 5 = -0.1079$.

The root lies between 2.2704 and 3. Let a = 2.2704 and b = 3.

Hence the fourth approximation is given by

$$x_4 = \frac{2.2704\times 13 - 3\times(-0.1079)}{13+0.1079} = 2.2764.$$

Now $f(x_4) = f(2.2764) = 11.7963 - 6.8292 - 5 = -0.0329$.

The root lies between 2.2764 and 3. Take a = 2.2764 and b = 3.

The fifth approximation is given by

$$x_5 = \frac{2.2764\times 13 - 3\times(-0.0329)}{13+0.0329} = 2.2782.$$

Now $f(x_5) = f(2.2782) = 11.8243 - 6.8346 - 5 = -0.0103$.

Since $f(3)$ is positive and $f(2.2782)$ is negative the root lies between 2.2782 and 3. Take a = 2.2782 and b = 3.

$\therefore$ The sixth approximation is given by

$$x_6 = \frac{2.2782 \times 13 - 3 \times (-0.0103)}{13 + 0.0103} = 2.2788.$$

Now $f(2.2788) = 11.8336 - 6.8364 - 5 = -0.0028$.

Since $f(x_6)$ is negative and $f(3)$ is positive, the root lies between 2.2788 and 3. Let $a = 2.2788$ and $b = 3$.

$$x_7 = \frac{2.2788 \times 13 - 3(-0.0028)}{13 + 0.0028} = 2.2790.$$

$\therefore x_6 \simeq x_7 = 2.279$ (corrected upto 3 places of decimals).

Problem 3. Find the smallest positive root of $x^2 - \log_e x - 12 = 0$ by Regula falsi method.

Solution. Let

$$f(x) = x^2 - \log_e x - 12$$

$$f(3) = 9 - 1.0986 - 12 = -4.0986$$

$$f(4) = 16 - 1.3863 - 12 = 2.6137.$$

$\therefore$ The root lies between 3 and 4. Let $a = 3$ and $b = 4$.

$\therefore$ The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{3\,f(4) - b\,f(3)}{f(4) - f(3)}$$

$$= \frac{3 \times 2.6137 - 4 \times (-4.0986)}{2.6137 - (-4.0986)} = 3.6106.$$

Now $f(x_1) = f(3.6106) = 13.0364 - 1.2839 - 12 = -0.2474$.

We note that $f(4)$ is positive and $f(3.6106)$ is negative.

$\therefore$ The root lies between 3.6106 and 4. Take $a = 3.6106$ and $b = 4$.

$\therefore$ The second approximation is

$$\therefore x_2 = \frac{3.6106 \times 2.6137 - 4 \times (-0.2474)}{2.6137 + 0.2474} = 3.6443.$$

Now $f(3.6443) = 13.2809 - 1.2932 - 12 = -0.0123$ the root lies between 4 and 3.6443. Hence take $a = 3.6443$ and $b = 4$.

$\therefore$ The third approximation is

$$\therefore x_3 = \frac{3.6443 \times 2.6137 - 4 \times (-0.0123)}{2.6137 - (-0.0123)} = 3.6460.$$

Now $f(3.6460) = 13.2933 - 1.2936 - 12 = -0.0003$ and the root lies between 4 and 3.646. Let $a = 3.646$ and $b = 4$.

$\therefore$ The fourth approximation is

$$\therefore x_4 = \frac{3.646 \times 2.6137 - 4 \times (-0.0003)}{2.6137 - (-0.0003)} = 3.6461.$$

Hence the required root is 3.646 (corrected upto 3 decimals).

Problem 4. Find by Regula falsi method the positive root of $x^2 - \log_{10} x - 12 = 0$.

Solution. Let

$$f(x) = x^2 - \log_{10} x - 12$$

$$f(3) = 9 - 0.4771 - 12 = -3.4771$$

$$f(4) = 16 - 0.6021 - 12 = 3.3979.$$

$\therefore$ The root lies between 3 and 4. Let $a = 3$ and $b = 4$.

$\therefore$ The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{3 \times 3.3979 - 4\,(-3.4771)}{3.3979 - 3.4771} = 3.5058.$$

Now $f(x_1) = f(3.5058) = (3.5058)^2 - \log_{10} 3.5058 - 12 = -0.2542$.

$\therefore$ The root lies between 3.5058 and 4.

By taking $a = 3.5058$ and $b = 4$ we get the second approximation as

$$x_2 = \frac{3.5058 \times 3.3979 - 4 \times (-0.2542)}{3.3979 - (-0.2542)} = 3.5402.$$

Now $f(x_2) = f(3.5402) = (3.5402)^2 - \log_{10} 3.5402 - 12 = -0.016.$

The root lies between $a = 3.5402$ and $b = 4$.

Hence the third approximation is

$$x_3 = \frac{3.5402 \times 3.3979 - 4 \times (-0.016)}{3.3979 - (-0.016)} = 3.542.$$

Now $f(x_3) = -0.0007.$

The root lies between $a = 3.542$ and $b = 4$.

$\therefore$ The fourth approximation is

$$\therefore x_4 = \frac{3.5424 \times 3.3979 - 4\,(-0.0007)}{3.3979 - (-0.0007)} = 3.542.$$

Hence the required root is 3.542.

Exercises

Solve the following equations using Regula Falsi method

1. $x^3 - 4x + 1 = 0$ which lies between 0 and 1
2. $2x - 3 \sin x = 5$

Algebraic and Transcendental Equations

Unit II

2.1Bisection Method (Bolzano method)

Let $f(x)$ be a continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite signs. Hence one root of the equation $f(x) = 0$ lies between a and b. For definiteness we assume that $f(a) < 0$ and $f(b) > 0$. **Bisection method** is used to find the root between a and b to the desired approximation as follows.

(i) Let $x_1 = \frac{a+b}{2}$ be the first approximation of the required root (x_1 is the *midpoint* of a and b).

(ii) If $f(x_1) = 0$ then x_1 is a root of $f(x)$. If not the root lies between a and x_1 or x_1 and b depending on whether $f(x_1) > 0$ or $f(x_1) < 0$.

(iii) Bisect the interval in which the root lies and continue the process until the root is found to the desired accuracy.

Problem 1. Find a real root of the equation $x^3 - 3x + 1 = 0$ lying between 1 and 2 correct to three places of decimal by using bisection method.

Solution. Let $f(x) = x^3 - 3x + 1$

Since $f(1) = -1$ and $f(2) = 3$, $f(x) = 0$ has one root lying between 1 and 2

Let $a = 1$ and $b = 2$.

The first approximation is

$$x_1 = \frac{a+b}{2} = 1.5.$$

Now $f(1.5) = -0.125$. Also $f(2)$ is positive. Hence the root lies between 1.5 and 2. Let $a = 1.5$ and $b = 2$.

$\therefore$ The second approximation is

$$x_2 = \frac{a+b}{2} = \frac{1.5+2}{2} = 1.75.$$

Now $f(1.75) = 1.1094$. Since $f(1.5)$ is negative and $f(1.75)$ is positive the root lies between 1.5 and 1.75. Let $a = 1.5$ and $b = 1.75$.

∴ The third approximation is

$$x_3 = \frac{1.5+1.75}{2} = 1.625$$

Now $f(1.625) = 1.625^3 - 3 \times 1.625 + 1 = 4.2910 - 4.4875 + 1 = 0.4160$.

Since $f(1.5)$ is negative and $f(1.625)$ is positive the root lies between 1.5 and 1.625.

This process is repeated and the calculation of the successive approximations is given in the following table.

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1	1	2	$x_1 = 1.5$	– 0.125
2	1.5	2	$x_2 = 1.75$	1.1094
3	1.5	1.75	$x_3 = 1.625$	0.8035
4	1.5	1.1625	$x_4 = 1.5625$	0.1272
5	1.5	1.5625	$x_5 = 1.5313$	– 0.0032
6	1.5313	1.5625	$x_6 = 1.5469$	0.0609
7	1.5313	1.5469	$x_7 = 1.5391$	0.0286
8	1.5313	1.5391	$x_8 = 1.5352$	0.0126
9	1.5313	1.5352	$x_9 = 1.5333$	0.0049
10	1.5313	1.5333	$x_{10} = 1.5323$	0.0009
11	1.5313	1.5323	$x_{11} = 1.5318$	–

We observe that $x_{10} = x_{11} = 1.532$ correct to 3 places of decimals.

Hence the required root, correct to three places of decimals is 1.532.

Problem 2. Find a real root of the equation $x^3 - x - 11 = 0$ by using bisection method.

Solution. Let $f(x) = x^3 - x - 11$.

$f(2) = -5$ and $f(3) = 13$.

∴ One root of $f(x) = 0$ lies between 2 and 3. Let $a = 2$ and $b = 3$.

The first approximation is $x_1 = \frac{a+b}{2} = 2.5$.

Now $f(2.5) = 2.125$ which is positive and $f(2)$ is negative.

Hence the root lies between 2 and 2.5.

Let $a = 2$ and $b = 2.5$

$\therefore$ The second approximation is $x_2 = \frac{2+2.5}{2} = 2.25$.

Now $f(2.25) = -1.8594$ and the root lies between 2.25 and 2.5.

This process is repeated and the calculation of the successive approximations in given in the following table.

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1	2	3	$x_1 = 2.5$	2.125
2	2	2.5	$x_2 = 2.25$	– 1.18594
3	2.25	2.5	$x_3 = 2.375$	0.0215
4	2.25	2.375	$x_4 = 2.3125$	– 0.9460
5	2.3125	2.375	$x_5 = 2.3438$	– 0.4684
6	2.3438	2.375	$x_6 = 2.3594$	– 0.2252
7	2.3594	2.375	$x_7 = 2.3672$	– 0.1023
8	2.3672	2.375	$x_8 = 2.3711$	– 0.0405
9	2.3711	2.375	$x_9 = 2.3731$	– 0.0087
10	2.3731	2.375	$x_{10} = 2.3741$	0.0072
11	2.3731	2.3741	$x_{11} = 2.3736$	– 0.0008
12	2.3736	2.3741	$x_{12} = 2.3739$	0.004
13	2.3736	2.3739	$x_{13} = 2.3738$	0.0024
14	2.3736	2.3738	$x_{14} = 2.3737$	0.0008
15	2.3736	2.3737	$x_{15} = 2.3737$	0.0008

From the table we see that $x_{10} = x_{11} = 2.374$, correct to three places of decimals and $x_{14} = x_{15} = 2.3737$. Hence the value of the root upto three places of decimals in 2.374 and upto four places of decimals is 2.3737.

2.2 Newton – Raphson method

Let x_0 be an approximate root of the equation $f(x) = 0$. Let $x_1 = x_0 + h$ be the exact root where h is very small, positive or negative.

$$\therefore f(x_0) = 0. \qquad \ldots\ldots (1)$$

By Taylor's series expansion, we have

$$f(x_1) = f(x_0 + h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots$$

Since $f(x_1) = 0$ and h is very small h^2 and higher powers of h can be neglected. Hence $f(x_0) + h f'(x_0) = 0$.

$$\therefore h = -\frac{f(x_0)}{f'(x_0)} \text{ if } f'(x_0) \neq 0.$$

Hence $x_1 = x - \dfrac{f(x_0)}{f'(x_0)}$ is a first approximation to the root.

Similarly starting with x_1 we get the next approximation to the root given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2, \ldots$$

and it is known as ***Newton – Raphson's iteration formula*** **or simply Newton Rapson's formula**.

Lemma: The order of convergence of the Newton Rapson's method is atleast two

Note. Newton – Raphson method is also referred as the *method of tangents*.

Problem 1. Find the first approximation of the root lying between 0 and 1 of the equation $x^3 + 3x - 1 = 0$ by Newton-Raphson formula.

Solution. Let $f(x) = x^3 + 3x - 1$. Hence $f'(x) = 3x^2 + 3$.

Given that the root lies between 0 and 1. Take $x_0 = 0$.

Newton's formula is $x_{x+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

When n = 0, the first approximation is $x_1 = x_0 - \dfrac{f(x_0)}{f(x_1)}$.

Now $f(x_0) = f(0) = -1$

$f(x_0) = f'(0) = 3$

$\therefore x_1 = 0 - \left(\dfrac{-1}{3}\right) = 0.3333.$

$\therefore$ The first approximation is 0.3333.

Problem 2. Write Newton – Raphson formula to obtain the cube root of N.

Solution. Let $x = \sqrt[3]{N}$.

$\therefore x^3 = N$. Hence $x^3 - N = 0$.

Let $f(x) = x^3 - N$ and hence $f'(x) = 3x^2$.

The Newton – Raphson formula is

$$x_{x+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \qquad n = 0, 1, 2\ldots\ldots$$

$$= x_n - \frac{(x_n^3 - N)}{3x_n^2}$$

Problem 3. Find the real root $x^3 - 3x + 1 = 0$ lying between 1 and 2 upto three decimal places by Newton-Raphson method.

Solution. Let $f(x) = x^3 - 3x + 1$. Hence $f'(x) = 3x^2 - 3$

$f(1) = -1$ and $f(2) = 3$. Hence one root lies between 1 and 2.

Let the initial approximation be $x_0 = 1$. The Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; n = 0, 1, 2 \ldots$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now $f(x_0) = f(1) = -1$; $f'(x_0) = f'(1) = 0$.

Since $f'(1) = 0$. Newton's approximation formula cannot be applied for the initial approximation as $x = 1$. Let us take $x_0 = 1.5$.

$$\text{Now} \quad f(x_0) = f(1.5) = 3.375 - 4.5 + 1 = -0.125$$

$$f'(x_0) = f'(1.5) = 6.75 - 3 = 3.75$$

$$\therefore x_1 = 1.5 - \left(\frac{-0.125}{3.75}\right) = 1.5 + 0.0333$$

$$\therefore x_1 = 1.5333$$

$$\text{Now,} \quad f(x_1) = (1.5333)^3 - 3 \times 1.5333 + 1 = .0049$$

$$\text{and} \quad f'(x_1) = 3 \times (1.5333)^2 - 3 = 4.053.$$

$$\text{Hence } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5333 - \left(\frac{0.0049}{4.053}\right) = 1.5321$$

$$\text{Now,} \quad f(x_2) = f(1.5321) = 3.5963 - 4.5963 + 1 = 0$$

$\therefore x_2$ is a root of $f(x) = 0$.

Hence the required root is 1.532 corrected upto 3 decimal places.

Problem 4. Find the real root of $xe^x - 2 = 0$ correct to three places of decimals using Newton – Raphson method.

Solution. Let $f(x) = xe^x - 2$

$$\therefore f'(x) = xe^x + e^x = e^x(x + 1).$$

We note that $f(0) = -2$ and $f(1) = e - 2 = 2.7183 - 2 = 0.7183$.

$\therefore$ The root lies between 0 and 1

Since the numerical value of $f(1)$ is less than that of $f(0)$ we can take the initial approximation as $x_0 = 1$.

The Newton – Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$; $\quad n = 0, 1, 2\ldots\ldots$

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now, $f(x_0) = f(1) = 0.7183$; $f'(x_0) = f'(1) = 2.7183\,(1 + 1) = 5.4366$

$$\therefore\ x_1 = 1 - \frac{0.7183}{5.4366} = 1 - 0.1321 = 0.8679$$

Now, $f(x_1) = 0.0673$ and $f'(x_1) = 4.4492$

$$\therefore\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8679 - \frac{0.0673}{4.4492} = 0.8679 - 0.0151 = 0.8528.$$

Now, $f(x_2) = f(0.8528) = 0.8528 \times e^{0.8528} - 2 = .0008$

$f'(x_2) = e^{.8528} \times 1.8528 = 4.3471$

$$\therefore\ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.8528 - \frac{0.0008}{4.3471} = 0.8526.$$

Since x_2 and x_3 are approximately equal, upto third decimals we can take the required root as 0.853.

The three iterations are given in the following table

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$	x_{n+1}
0	1	0.7183	5.4366	0.1321	0.8679	x_1
1	0.8679	0.0673	4.4492	0.0151	0.8528	x_2
2	0.8528	0.0008	4.3470	0.0002	0.8526	x_3

Problem 5. Evaluate $\sqrt{12}$ to four places of decimals by Newton-Raphson Method.

Solution. Let $x = \sqrt{12}$

$\therefore\ x^2 = 12.$ Hence $x^2 - 12 = 0.$

Let $f(x) = x^2 - 12$. Hence $f'(x) = 2x.$

We note $f(3) = -3$ and $f(4) = 4$.

$\therefore$ The root lies between 3 and 4.

Since the numerical value of f (3) is less than that of f (4) we take the initial approximation as $x_0 = 3$.

The Newton – Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now, $f(x_0) = f(3) = 9 - 12 = -3;\ f'(x_0) = f'(3) = 6$

$$\therefore\ x_1 = 3 - \left(\frac{-3}{6}\right) = 3 + 0.5 = 3.5$$

Now, $f(x_1) = f(3.5) = 12.25 - 12 = 0.25$

$f'(x_1) = f'(3.5) = 7.$

$$\therefore\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.5 - \left(\frac{0.25}{7}\right) = 3.4643.$$

Now, $f(x_2) = f(3.4643) = 12.0014 - 12 = .0014$

$f'(x_2) = f'(3.4643) = 2 \times 3.4643 = 6.9286$

$$\therefore\ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.4643 - \left(\frac{0.0014}{6.9286}\right) = 3.4641.$$

$\therefore$ The required root is 3.4641 to 4 places of decimals.

Problem 6. Using Newton-Raphson iterative method find the real root $x\log_{10} x = 1.2$ correct to four decimal places.

Solution. Let $f(x) = x\log_{10} x - 1.2$

$$\therefore f'(x) = x\left(\frac{\log_{10} e}{x}\right) + \log_{10} x = \log_{10} e + \log_{10} x$$

$$= 0.4343 + \log_{10} x$$

$f(1) = -1.2$; $f(2) = 2 \times 0.3010 - 1.2 = 0.6020 - 1.2 = -0.598$

$f(3) = 3 \times 0.4771 - 1.2 = 1.4313 - 1.2 = 0.2313$

$\therefore$ One root lies between 2 and 3. Let the initial approximation be $x_0 = 2$.

Newton – Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$; $n = 0, 1, 2 \ldots$

When $n = 0$ the first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now, $f(x_0) = f(2) = 0.6020 - 1.2 = -0.598$

$f'(x_0) = f'(2) = 0.4343 + 0.3010 = 0.7353$

$$\therefore x_1 = 2 - \frac{(-0.598)}{0.7353} = 2 + 0.8133 = 2.8133$$

The second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8133 - \frac{f(2.8133)}{f'(2.8133)}$$

Now, $f(2.8133) = 2.8133 \times 0.4492 - 1.2 = 1.2637 - 1.2 = 0.0637$

$f'(2.8133) = 0.4343 + 0.4492 = 0.8835$

$$\therefore \quad x_2 = 2.8133 - \left(\frac{0.0637}{0.8835}\right) = 2.8133 - 0.0721 = 2.7412$$

The third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7412 - \frac{f(2.7412)}{f'(2.7412)}$$

Now, $f(2.7412) = 2.7412 \times 0.4379 - 1.2$

$= 1.2004 - 1.2 = .0004$

$f'(2.7412) = 0.4343 + 0.4379 = 0.8722$

$\therefore\ x_3 = 2.7412 - \dfrac{.0004}{0.8722} = 2.7412 - 0.0005 = 2.7407$

$\therefore\ x_2 \approx x_3 = 2.741$ (correct to 3 decimal places)

$\therefore$ The required root is 2.741.

Problem 7. Find by Newton-Raphson method correct to 4 places of decimals the root between 0 and 1 of the equation $3x - \cos x - 1 = 0$.

Solution: Let $f(x) = 3x - \cos x - 1$. Hence $f'(x) = 3 + \sin x$.

Now $f(0) = -2$; $f(1) = 3 - 0.5403 - 1 = 1.4597$.

Since the numerical value of $f(1)$ is less than that of $f(0)$ we can take the initial approximation as $x_0 = 1$.

The Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \qquad n = 0, 1, 2, \ldots..$$

$\therefore$ The first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(x_0) = f(1) = 1.4597; \qquad f'(x_0) = f'(1) = 3 + 0.8415 = 3.8415$

$\therefore\ x_2 = 0.62 - \dfrac{0.0461}{3.581} = 0.6071$

The third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Now, $f(x_2) = f(0.6071) = 0;\ f'(x_2) = f'(0.6071) = 3.5705$

$\therefore\ x_3 = 0.6071 - \left(\dfrac{0}{3.5705}\right) = 0.6071$

Since x_3 and x_4 are equal upto 4 decimal places the required root is 0.6071.

Exercise

1.Find the positive root of each of the following equations using Newton Raphson method

(i) xsinx + cosx = 0
(ii) $2x^3 - 3x - 6 = 0$
(iii) $x^4 + x - 10 = 0$

2.3 Horner's method

Horner's method is the most convenient way of finding approximate values of the irrational roots of the equation $f(x) = 0$ where $f(x)$ is a any polynomial. The root is calculated in decimal form and the figures of the decimal are obtained in succession. We describe below the steps to be followed.

Step I. Consider the equation $f(x) = 0$. Suppose this has a single root α in the interval $(a, a + 1)$ where a is a positive inter. Then a can be located by using the condition that $f(a)$ and $f(a + 1)$ are of opposite signs.

Step II. Suppose the exact value of the root is $a.a_1a_2\ldots\ldots$ Diminish the roots of $f(x) = 0$ by a. Then we get the transformed equation $f_1(x) = 0$ having $0.a_1a_2\ldots\ldots$ as a root.

Step III. Multiply the roots of $f_1(x) = 0$ by 10 and we obtain the transformed equation $f_2(x) = 0$ having $a_1.a_2a_3\ldots\ldots$ as a root.

Step IV. By inspection we locate the root by finding two consecutive integers b and $b + 1$ such that $f_2(b)$ and $f_2(b + 1)$ are of opposite signs. Then $b = a_1$ is the first decimal in the root making $a.a_1$ as the first approximation of the root.

Repeat this process (Step I to IV) as many times as needed to get the roots of $f(x) = 0$ to any desired number of decimal places.

Problem 1. Show that the equation $x^3 - 3x + 1 = 0$ has a root between 1 and 2 and calculate it to three decimal places by Horner's method.

Solution. Step 1. Let $f(x) = x^3 - 3x + 1$

We note, $f(1) = -1$ and $f(2) = 3$

$\therefore$ One root lies between 1 and 2.

Let the root be $1.a_1a_2$ ......

Step 2. Diminish the root of $f(x) = 0$ by 1

1	1	0	– 3	1
		1	1	– 2
	1	1	– 2	**– 1**
		1	2	
	1	2	**0**	
		1		
	1	**3**		

$\therefore$ The transformed equation is $f_1(x) = x^3 + 3x^2 - 1$ and $f_1(x) = 0$ has $0.a_1a_2$ ...... as a root.

Step 3. Multiplying the roots of the equation $f_1(x) = 0$ by 10 we get the transformed equation as

$$f_2(x) = x^3 + 30x^2 - 1000 = 0.$$

Which has a root $a_1.a_2a_3$ ......

Step 4. We note that $f_2(5) = -125 < 0$ and $f_2(6) = 296 > 0$.

$\therefore$ Hence $a_1 = 5$ and upto first approximation the root is 1.5.

Step 5. Diminish the roots of $f_2(x) = 0$ by 5.

5	1	30	0	– 1000
		5	175	875
	1	35	175	**– 125**
		5	200	
	1	40	**375**	
		5		
	1	**45**		

$\therefore$ The transformed equation is

$f_3(x) = x^3 + 45x^2 + 375x - 125.$

Step 6. Multiplying the roots of $f_3(x) = 0$ by 10 and we get

$$f_4(x) = x^3 + 450x^2 + 37500x - 125000 = 0.$$

and now the root is $a_2.a_3a_4......$

Step 7. $f_4(3) = -8423 < 0$; $f_4(4) = 32264 > 0$

$\therefore\ a_2 = 3$ and upto second approximation of the root is 1.53

Step 8. Diminish the roots of $f_4(x) = 0$ by 3

3	1	450	37500	– 125000
		3	1359	116577
	1	453	38859	**– 8423**
		3	1368	
	1	456	**40227**	
		3		
	1	**459**		

$\therefore$ The transformed equation is

$$f_5(x) = x^3 + 459x^2 + 40227x - 8423.$$

Step 9. Multiplying the roots of $f_5(x) = 0$ by 10 and we get

$$f_6(x) = x^3 + 4590x^2 + 4022700x - 8423000$$

and now the root is $a_3.a_4a_5......$

Step 10.

$$a_3 = \frac{8423000}{4022700} \quad \text{(Neglecting higher powers } x\text{)}$$

$$= 2.0939.$$

$\therefore\ a_3 = 2$ and hence upto the third approximation the root is 1.532.

Problem 2. Find the positive root of $x^3 - x - 3 = 0$ correct to two places of decimal places by Horner's method.

Solution. Step 1. Let $f(x) = x^3 - x - 3$

We note, $f(1) = -3 < 0$ and $f(2) = 3 > 0$

$\therefore$ One root lies between 1 and 2.

Let the root be $1.a_1a_2 ......$

Step 2. Diminish the root of $f(x) = 0$ by 1

1	1	0	– 1	-3
		1	1	0
	1	1	0	**– 3**
		1	2	
	1	2	**2**	
		1		
	1	**3**		

$\therefore$ The transformed equation is $f_1(x) = x^3 + 3x^2 + 2x - 3 = 0$ and it has $0.a_1a_2 ......$ as a root.

Step 3. Multiplying the roots of the equation $f_1(x) = 0$ by 10 we get the transformed equation as

$$f_2(x) = x^3 + 30x^2 + 200x - 3000 = 0.$$

Which has a root $a_1.a_2a_3 ......$

Step 4. We note that $f_2(6) = -504 < 0$ and $f_2(7) = 213 > 0$.

$\therefore$ Hence $a_1 = 6$ and upto first approximation the root is 1.6.

Step V. Diminish the roots of $f_2(x) = 0$ by 6.

6	1	30	200	– 3000
		6	216	2496
	1	36	416	**– 504**
		6	252	

1	42	**668**
	5	
1	**48**	

∴ The transformed equation is

$$f_3(x) = x^3 + 48x^2 + 668x - 504.$$

Step 6. Multiplying the roots of $f_3(x) = 0$ by 10 and we get

$$f_4(x) = x^3 + 480x^2 + 66800x - 504000 = 0.$$

and now the root is $a_2.a_3a_4......$

Step 7. $f_4(7) = -12537 < 0$; $f_4(4) = 61632 > 0$

∴ $a_2 = 7$ and upto second approximation of the root is 1.67

Step 8. Diminish the roots of $f_4(x) = 0$ by 7

7	1	480	66800	– 504000
		7	3409	491463
	1	487	70209	**– 12537**
		7	1368	
	1	494	**73667**	
		7		
	1	**501**		

∴ The transformed equation is

$$f_5(x) = x^3 + 501x^2 + 73667x - 12537.$$

Step 9. Multiplying the roots of $f_5(x) = 0$ by 10 and we get

$$f_6(x) = x^3 + 5010x^2 + 7366700x - 12537000$$

and now the root is $a_3.a_4a_5......$

Step 10.

$$a_3 = -\left(\frac{-12537000}{7366700}\right) \quad \text{(Neglecting higher powers } x\text{)}$$

$$= 1.70.$$

$\therefore$ $a_3 = 1$ and hence upto the third approximation the root is 1.671 and correct to 2 places of decimals, it is 1.67.

Exercises

1. Find by Horner's method the root of the equation $x^3 - 4x^2 + 5 = 0$ which lies between 1 and 2 to 2 places of decimals.
2. Find by Horner's method the root of the equation $x^3 + 3x - 1 = 0$ correct to 2 places of decimals.
3. Find the negative root of $x^3 - x^2 + 12x + 24 = 0$ correct to 2 places of decimals by using Horner's Method.

Simultaneous Equations

Unit III

3. Introduction

A system of m linear equation in n unknowns $x_1, x_2, ..., x_n$ is given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots &= \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

This set of equations can be written in the matrix form $A\,X = B$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}; X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The $m \times n$ matrix A is called the *coefficient matrix*.

The $m \times (n + 1)$ matrix given by

$$(A, B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right)$$

is called the *augmented* matrix of the system.

3.1 Back Substitution

Consider a system of simultaneous linear equations given by $AX = B$ where A is an $n \times n$ coefficient matrix.

Suppose the matrix A is upper triangular.

Let $\quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{mn} \end{pmatrix}$

Then the given system takes the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33}\cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0\cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

(i.e.)
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \ldots + a_{1n}x_n &= b_2 \\ \ldots \quad \ldots \quad \ldots &= \ldots \\ a_{n-1n-1}x_{n-1} + \ldots + a_{n-1}x_n &= b_{n-1} \\ a_{nn}x_n &= b_n \end{aligned}$$

From the last equation we get $x_n = \dfrac{b_n}{a_{nn}}$.

Proceeding like this we can find all x_i 's. This procedure is known as *back substitution*.

Similarly considering lower triangular matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

the given system takes the form

$$\begin{aligned} a_{11}x_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \\ \ldots \quad \ldots \quad & \ldots. \end{aligned}$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n \quad = \quad b_n$$

From the first equation we get $x_1 = \dfrac{b_1}{a_{11}}$

Proceeding like this we can find all x_i's. This procedure is known as *forward substitution*.

3.2 Gauss Elimination Method

Gauss elimination method is a direct method which consists of transforming the given system of simultaneous equations to an equivalent *upper triangular system*.

The row operation

$$R_i \to R_{i-1} - \frac{a_{ik}}{a_{kk}} R_k \; ; \; i = k+1,\, k+2,\, \ldots, n$$

will make all the entries $a_{k+1,k,}\, a_{k+2,k,}\, \ldots, a_{nk}$ in the k^{th} column zero.

Hence the given system of equations is reduced to the form $UX = D$ where U is an upper triangular matrix. The required solution can be obtained by the method of *back substitution*.

Problem 1 Solve the equations $x + y = 2$ and $2x + 3y = 5$ by Gauss elimination method.

Solution: The given set of equations can be written as

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{matrix} 2 \\ 5 \end{matrix}$$

The augmented matrix is

$$(A, B) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array}\right)$$

We note $a_{11} = 1 \neq 0$ is the pivot. The first equation is the pivot equation and $-\left(\dfrac{2}{1}\right)$ is the multiplier for the second equation.

$$(A, B) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right) \begin{matrix} R_1 \to R_1 \\ R_2 \to R_2 - 2R_1 \end{matrix}$$

The given set of equation is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{matrix} 2 \\ 1 \end{matrix}$

$\therefore x + y = 2$ and $y = 1$.

By back substitution we get $y = 1$ and $x = 1$.

Problem 2. Solve the following system of equations using Gaussian elimination method

$$\begin{aligned} x + y + z &= 9 \\ 2x - 3y + 4z &= 13 \\ 3x + 4y + 5z &= 40 \end{aligned}$$

Solution. The given set of equations can be written as

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix}$$

The augmented matrix is $(A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right)$

$$(A, B) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & \frac{12}{5} & 13 \end{array}\right) \quad R_3 \to R_3 + \frac{1}{5} R_2$$

$\therefore$ The given system of equation reduces to the system

$$\begin{aligned} \frac{12}{5} z &= 12 \\ -5y + 2z &= -5 \\ \text{and } x + y + z &= 9 \end{aligned}$$

Now by back substitution we obtain the solution $x = 1$; $y = 3$; $z = 5$.

Exercises

Solve the following system of equations by Gauss Elimination Method

1. $2x + 3y = 5;\ \ 3x - y = 2$
2. $x - y + z = 1;\ \ -3x + 2y - 3z = -6;\ \ 2x - 5y + 4z = 5$

3.3 Gauss-Jordan Elimination Method

Consider the system of equations $AX = B$

If A is a *diagonal matrix* the given system reduces to

$$\begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ 0 & a_{22} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The system reduces to the following n equations

$$a_{11}x_1 = b_1; \qquad a_{22}x_2 = b_2; \qquad a_{nn}x_n = b_n;$$

Hence we get the solution directly as $x_1 = \dfrac{b_1}{a_{11}};\ x_2 = \dfrac{b_2}{a_{22}} \ldots x_n = \dfrac{b_n}{a_{nn}}$

The method of obtaining the solution of the system of equation by reducing the matrix A to a diagonal matrix is known as **Gauss-Jordan elimination** method.

Problem 3: Solve the following equations by Gauss Jordan method.

$$x + y = 2$$

$$2x + 3y = 5$$

Solution: The augmented matrix is $(A, B) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array}\right)$

$$(A, B) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right) \quad R_2 \to R_2 - 2R_1$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right) \quad R_1 \to R_1 - R_2$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore$ $x = 1$; $y = 1$.

Problem 4. Solve the following system of equations by Gauss Jordan method:

$$\begin{aligned} x + y + z &= 9 \\ 2x - 3y + 4z &= 13 \\ 3x + 4y + 5z &= 40 \end{aligned}$$

Solution. The augmented matrix is $(A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right)$

$$(A, B) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 + (-3)R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{7}{5} & 8 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & \frac{12}{5} & 12 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 + \frac{1}{5}R_2 \\ R_3 \to R_3 + \frac{1}{5}R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & -5 & 0 & -15 \\ 0 & 0 & \frac{12}{5} & 12 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 + \left(-\frac{7}{12}\right)R_3 \\ R_2 \to R_2 + \left(-\frac{-10}{12}\right)R_3 \end{array}$$

$\therefore$ The given system reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & \frac{12}{5} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -15 \\ 12 \end{pmatrix}$$

$$\therefore\ x = 1; \qquad -5y = -15; \qquad \frac{12}{5}z = 12.$$

Hence $x = 1;\ \ y = 3;\ \ z = 5.$

Exercises

Solve the following system of equations by Gauss Jordan Method

1. 10x + y + z = 12 ; 2x + 10y + z = 13; x + y + 5z = 7
2. 8x - 3y + 2z = 20 ; 2x + y + 4z = 12; 4x + 11y - z =33

3.4 Calculation of inverse of a matrix

$$\text{Let } X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

be the inverse of A.

$\therefore\ AX = I$ where I is the unit matrix of order n.

$\therefore\ AX = I$ gives

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0\cdots & 1 \end{pmatrix}$$

This equation is equivalent to the following n system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

……. ……. …….

……. ……. …….

and

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Each of the system of the above n systems of equations can be solved by Gauss elimination method or Gauss Jordan method.

Problem 5. Find the inverse of the matrix A = $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ using Gaussian method.

Solution. Let $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A.

$\therefore\ AX = I_3$ where I_3 is the 3×3 identity matrix.

The augmented system $\mathcal{A}$ can be written as

$$\mathcal{A} = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & \frac{7}{2} & \frac{17}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - \frac{3}{2}R_1 \\ R_3 \to R_3 - \frac{1}{2}R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right) \quad R_3 \to R_3 - 7R_2$$

The equation $AX = I_3$ is equivalent to the following three systems

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 10 \end{pmatrix} \quad \ldots (1)$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -7 \end{pmatrix} \quad \ldots (2)$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \ldots (3)$$

From (1) we get

$$\begin{aligned} 2x_{11} + x_{21} + x_{31} &= 1 \\ \frac{1}{2}x_{22} + \frac{3}{2}x_{31} &= -\frac{3}{2} \\ -2x_{31} &= 10. \end{aligned}$$

By backward substitution we get

$$x_{31} = -5; \qquad x_{21} = 12; \qquad x_{11} = -3 \qquad \ldots (4)$$

From (2) we get

$$\begin{aligned} 2x_{13} + x_{23} + x_{33} &= 0 \\ \frac{1}{2}x_{23} + \frac{3}{2}x_{33} &= 1 \\ -2x_{33} &= -7 \end{aligned}$$

By backward substitution we get

$$x_{32} = \frac{7}{2}; \qquad x_{22} = \frac{-17}{2}; \qquad x_{12} = \frac{5}{2} \qquad \ldots (5)$$

From (3) we get

$$2x_{12}+x_{22}+x_{32} = 0$$

$$\frac{1}{2}x_{22}+\frac{3}{2}x_{31} = 0$$

$$-2x_{31} = 1$$

By backward substitution we get

$$x_{33}=-\frac{1}{2}; \qquad x_{23}=\frac{3}{2}; \qquad x_{13}=\frac{-1}{2} \qquad \ldots (6)$$

From (4), (5), (6) we get the inverse of A as

$$X = \begin{pmatrix} -3 & \frac{5}{2} & \frac{-1}{2} \\ 12 & -\frac{17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & \frac{-1}{2} \end{pmatrix}$$

Exercises

Find by Gaussian Elimination the inverse of the matrix

1. $A = \begin{pmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{pmatrix}$

Simultaneous Equations

Unit IV

4.1 Iterative methods-Gauss Jacobi iteration method

Definition: An $n \times n$ matrix A is said to be **diagonally dominant** if the absolute value of each leading element is greater than or equal to the sum of the absolute values of the remaining elements in that row.

For example

$$A = \begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$$

is a diagonally dominant matrix a

$$B = \begin{pmatrix} 2 & 3 & -1 \\ 5 & 8 & -4 \\ 1 & 1 & 1 \end{pmatrix}$$

is not a diagonally dominant matrix.

Problem 1: Check whether the system of equations

$x + 6y - 2z = 5$

$4x + y + z = 6$

$-3x + y + 7z = 5$

is a diagonal system. If not make it a diagonal system.

Solution: The coefficient matrix

$$A = \begin{pmatrix} 1 & 6 & -2 \\ 4 & 1 & 1 \\ -3 & 1 & 7 \end{pmatrix}$$

is not diagonally dominant as such. However effecting the operations $C_2 \leftrightarrow C_1$ (inter changing column one and two) we get

$$A \sim \begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 1 \\ 1 & -3 & 7 \end{pmatrix}$$

which is a diagonally dominant matrix and the system is a diagonal system.

The system of equations can be written as

$$\begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 1 \\ 1 & -3 & 7 \end{pmatrix}\begin{pmatrix} y \\ x \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$$

Problem 2: Is the system of equations diagonally dominant? If not make it diagonally dominant.

3x + 9y - 2z = 10; 4x + 2y + 13z = 19; 4x - 2y + z =3

Solution: Consider the coefficient matrix

$$A = \begin{pmatrix} 3 & 9 & -2 \\ 4 & 2 & 13 \\ 4 & -2 & 1 \end{pmatrix}$$

Obviously A is not diagonally dominant as it is.

Now, $$A \sim \begin{pmatrix} 4 & 2 & 13 \\ 3 & 9 & -2 \\ 4 & -2 & 1 \end{pmatrix} R_2 \leftrightarrow R_1$$

$$\sim \begin{pmatrix} 13 & 2 & 4 \\ -2 & 9 & 3 \\ 1 & -2 & 4 \end{pmatrix} C_3 \leftrightarrow C_1$$

This is a diagonally dominant matrix and the system of equations can be written as

$$\begin{pmatrix} 13 & 2 & 4 \\ -2 & 9 & 3 \\ 1 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 19 \\ 10 \\ 3 \end{pmatrix}$$

Exercises

Check whether the following system of equations is a diagonal system. If not make it a diagonal system by rearranging the equations

1. x + 5y - z = 10 ; x + y + 8z = 20; 4x + 2y + z = 14
2. 2x - y + z = 2 ; x + y + 3z = 5; x + y + z =3

4.2 Gauss Jacobi iteration method

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n$$

We assume that the coefficient matrix of this system is diagonally dominant. The above equations can be rewritten as

$$x_1 = \frac{1}{a_{11}}\left(c_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n\right) \qquad \ldots (1)$$

$$x_2 = \frac{1}{a_{22}}\left(c_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right) \qquad \ldots (2)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_n = \frac{1}{a_{nn}}\left(c_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}\right) \qquad \ldots (n)$$

We start with the initial values for the variables $x_1, x_{2,} x_3, \cdots, x_n$ to be $x_1^{(0)}, x_2^{(0)}{}_{,} x_3^{(0)}, \cdots, x_n^{(0)}$

Using these values in (1), (2),…. , (n) respectively we get $x_1^{(1)}, x_2^{(1)}{}_{,} \cdots, x_n^{(1)}$.

Putting $x_1 = x_1^{(1)},\ x_2 = x_2^{(1)}{}_{,} \cdots, x_n = x_n^{(1)}$ in (1), (2), … , (n) respectively we get the next approximation $x_1^{(2)}, x_2^{(2)}{}_{,} \cdots, x_n^{(2)}$

In the general if the values of $x_1, x_2, \cdots, x_n$ in the r[th] iteration are $x_1^{(r)}, x_2^{(r)}{}_{,} \cdots, x_n^{(r)}$ then

$$x_1^{(r+1)} = \frac{1}{a_{11}}\left[c_1 - a_{12}x_1^{(r)} - a_{13}x_2^{(r)} - \cdots - a_{1n}x_n^{(r)}\right]$$

$$x_2^{(r+1)} = \frac{1}{a_{22}}\left[c_2 - a_{21}x_1^{(r)} - a_{23}x_2^{(r)} - \cdots - a_{1n}x_n^{(r)}\right]$$

$$x_n^{(r+1)} = \frac{1}{a_{nn}}\left[c_n - a_{n1}x_1^{(r)} - a_{n2}x_2^{(r)} - \cdots - a_{n,n-1}x_n^{(r)}\right]$$

Problem 3. Solve the following equations using Jacobi's iteration method.

$3x + 4y + 15z = 54.8$; $x + 12y + 3z = 39.66$; $10x + y - 2z = 7.74$.

Solution: Coefficient matrix of the given system of equations is

$$A = \begin{pmatrix} 3 & 4 & 15 \\ 1 & 12 & 3 \\ 10 & 1 & -2 \end{pmatrix}$$

We note that A is not diagonally dominant.

Also $A \sim \begin{pmatrix} 10 & 1 & -2 \\ 1 & 12 & 3 \\ 3 & 4 & 15 \end{pmatrix} \quad R_1 \leftrightarrow R_3$

which is diagonally dominant.

The given system becomes $10x + y - 2z = 7.74$ … (1)

$x + 12y + 3z = 39.66$ … (2)

$3x + 4y + 15z = 54.8$ … (3)

From (1), (2), (3) we get $x = \frac{1}{10}[7.74 - y + 2z]$ … (4)

$y = \frac{1}{12}[39.66 - x - 3z]$ … (5)

$z = \frac{1}{15}[54.8 - 3x - 4y]$ …

(6)

First iteration. Let the initial value be $x_0 = y_0 = z_0 = 0$

$x_1 = \frac{1}{10}[7.74] = 0.774$

$$y_1 = \frac{1}{12}[39.66] \quad = \quad 3.305$$

$$z_1 = \frac{1}{15}[54.8] \quad = \quad 3.6533$$

Second iteration

$$x_2 = \frac{1}{10}(7.74 - y_1 + 2z_1) = \frac{1}{10}(7.74 - 3.305 + 7.3066) = 1.1742$$

$$y_2 = \frac{1}{12}(39.66 - x_1 - 3z_1) = \frac{1}{12}(39.66 - 0.774 - 10.9599) = 2.3272$$

$$z_2 = \frac{1}{15}(54.8 - 3x_1 - 4y_1) = \frac{1}{15}(54.8 - 2.322 - 13.22) = 2.6172$$

Third iteration

$$x_3 = \frac{1}{10}(7.74 - y_2 + 2z_2) = \frac{1}{10}(7.74 - 2.3272 + 5.2344) = 1.0647$$

$$y_3 = \frac{1}{12}(39.66 - x_2 - 3z_2) = \frac{1}{12}(39.66 - 1.1742 - 7.8516) = 2.5529$$

$$z_3 = \frac{1}{15}(54.8 - 3x_2 - 4y_2) = \frac{1}{15}(54.8 - 3.5226 - 9.3088) = 2.7979$$

Fourth iteration

$$x_4 = \frac{1}{10}(7.74 - y_3 + 2z_3) = \frac{1}{10}(7.74 - 2.5529 + 5.5958) = 1.0783$$

$$y_4 = \frac{1}{12}(39.66 - x_3 - 3z_3) = \frac{1}{12}(39.66 - 1.0647 - 8.3937) = 2.5168$$

$$z_4 = \frac{1}{15}(54.8 - 3x_3 - 4y_3) = \frac{1}{15}(54.8 - 3.1941 - 10.2116) = 2.7596$$

Fifth iteration

$$x_5 = \frac{1}{10}(7.74 - y_4 + 2z_4) = \frac{1}{10}(7.74 - 2.5168 + 5.5192) = 1.0742$$

$$y_5 = \frac{1}{12}(39.66 - x_4 - 3z_4) = \frac{1}{12}(39.66 - 1.0783 - 8.2788) = 2.5252$$

$$z_5 = \frac{1}{15}(54.8 - 3x_4 - 4y_4) = \frac{1}{15}(54.8 - 3.2349 - 10.0672) = 2.7665$$

Sixth iteration

$$x_6 = \frac{1}{10}(7.74 - y_5 + 2z_5) = \frac{1}{10}(7.74 - 2.5252 + 5.533) = 1.0748$$

$$y_6 = \frac{1}{12}(39.66 - x_5 - 3z_5) = \frac{1}{12}(39.66 - 1.0742 - 8.2995) = 2.5239$$

$$z_6 = \frac{1}{15}(54.8 - 3x_5 - 4y_5) = \frac{1}{15}(54.8 - 3.2226 - 10.1008) = 2.7651$$

Seventh iteration

$$x_7 = \frac{1}{10}(7.74 - y_6 + 2z_6) = \frac{1}{10}(7.74 - 2.5239 + 5.5302) = 1.0746$$

$$y_7 = \frac{1}{12}(39.66 - x_6 - 3z_6) = \frac{1}{12}(39.66 - 1.0748 - 8.2953) = 2.5242$$

$$z_7 = \frac{1}{15}(54.8 - 3x_6 - 4y_6) = \frac{1}{15}(54.8 - 3.2244 - 10.0956) = 2.7653$$

After 7 iterations the difference in 6^{th} and 7^{th} iterations is very negligible.

Hence the solution of the system is given by x = 1.075; y = 2.524; z = 2.765 correct to three places of decimals.

Exercises

Solve the following system of equations by Gauss Jacobi iteration Method

1. 27x + 6y - z = 85 ; 6x + 15y + 2z = 72; x + y + 54z =110
2. x + 17y - 2z = 48 ; 2x + 2y + 18z = 30; 30x - 2y + 3z =48

4.3 Gauss-Seidel iteration method

Gauss-Seidel iteration method is a refinement of Gauss-Jacobi method. As in the Jacobi iteration method let

$$x_1 = \frac{1}{a_{11}}(c_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \qquad \ldots (1)$$

$$x_2 = \frac{1}{a_{22}}\left(c_2 - a_{12}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right) \quad \ldots (2)$$

… … … … …

$$x_n = \frac{1}{a_{nn}}\left(c_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}\right) \quad \ldots (n)$$

We start with the initial values $x_2^{(0)}, x_3^{(0)}, \cdots, x_n^{(0)}$ and we get from (1)

$$x_1^{(1)} = \frac{1}{a_{11}}\left[c_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \cdots - a_{1n}x_n^{(0)}\right]$$

In the second equation we use $x_1^{(1)}$ for x_1 and $x_3^{(0)}$ for x_3 etc and $x_n^{(0)}$ for x_n.

(In the Jacobi method were we use $x_1^{(0)}$ for x_1). Thus we get

$$x_2^{(1)} = \frac{1}{a_{22}}\left[c_2 - a_{11}x_1^{(1)} - a_{13}x_3^{(0)} - \cdots - a_{2n}x_n^{(0)}\right]$$

Proceeding like this we find the first iteration values as

$$x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}$$

In general if the values of the variables in the r[th] iteration are $x_1^{(r)}, x_2^{(r)}, \cdots, x_n^{(r)}$ then the values in the $(r+1)^{th}$ iteration are given by

$$x_1^{(r+1)} = \frac{1}{a_{11}}\left[c_1 - a_{12}x_2^{(r)} - a_{13}x_3^{(r)} - \cdots - a_{1n}x_n^{(r)}\right]$$

$$x_2^{(r+1)} = \frac{1}{a_{22}}\left[c_2 - a_{11}x_1^{(r+1)} - a_{13}x_3^{(r)} - \cdots - a_{1n}x_n^{(r)}\right]$$

$$x_n^{(r+1)} = \frac{1}{a_{nn}}\left[c_n - a_{n1}x_2^{(r+1)} - a_{n2}x_{23}^{(r+1)} - \cdots - a_{n,n-1}x_{n-1}^{(r+1)}\right]$$

Problem 1: Solve 2x+y = 3; 2x+ 3y = 5 by Gauss Seidel iteration method.

Solution: Clearly the coefficient matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ is diagonally dominant and hence Gauss Seidel iteration method can be applied.

The given equations can be written as

$$x = \frac{1}{2}(3 - y) \qquad \text{-------- (1)}$$

$$y = \frac{1}{2}(5 - 2x) \qquad \text{-------- (2)}$$

Putting y = 0 in (1) we get x = 1.5

Using x = 1.5 in (2) we get y = 0.6667

Putting y = 0.6667 in (1) we get x = 1.1667

Putting x = 1.1667 in (2) we get y =0.8889

Putting y =0.8889 in (1) we get x = 1.0556

Putting x = 1.0556 in (2) we get y = 0.9629

Putting y = 0.9629 in (1) we get x = 1.0186

Putting x = 1.0186 in (2) we get y = 0.9876

Putting y = 0.9876 in (1) we get x = 1.0062

Putting x = 1.0062 in (2) we get y = 0.9959

Using y = 0.9959 in (1) we get x = 1.0021

Using x = 1.0021 in (2) we get y = 0.9986

Using y = 0.9986 in (1) we get x = 1.0007

Putting x = 1.0007 in (2) we get y = 0.9995

Putting y = 0.9995 in (1) we get x = 1.0001

Putting x = 1.0001 in (2) we get y = 0.9999

Using y = 0.9999 in (1) we get x = 1

Using x= 1 in (2) we get y = 1

Hence x = 1, y = 1 is the solution of the two equations.

Note: The above iterations can be simply carried out and exhibited in the following tabular form.

Iteration	Start	1	2	3	4	5	6	7	8	9	10
x	--	1.5	1.6667	1.0556	1.0186	1.0062	1.0021	1.0007	1.0001	1	1
y	0	0.6667	0.8889	0.9629	0.9876	0.9959	0.9986	0.9995	0.9999	1	1

Problem 2. Solve the following system of equation using Gauss Seidel iteration method.

$6x + 15y + 2z = 72$; $x + y + 54z = 110$; $27x + 6y - z = 85$

Solution: Coefficient matrix of the given system of equation is

$$A = \begin{pmatrix} 6 & 15 & 2 \\ 1 & 1 & 54 \\ 27 & 6 & -1 \end{pmatrix}$$

We note that A is not diagonally dominant.

However it can be made diagonally dominant by changing the rows as

$$A = \begin{pmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{pmatrix}$$

Hence the corresponding system of equation is

$27x + 6y - z = 85$

$6x + 15y + 2z = 72$

$x + y + 54z = 110$

The above system of equations can be rewritten as

$$x = \frac{1}{27}(85 - 6y + z) \quad \ldots (1)$$

$$y = \frac{1}{15}(72 - 6x - 2z) \quad \ldots (2)$$

$$z = \frac{1}{54}(10 - -x - y) \quad \ldots (3)$$

First iteration

Putting $y = 0$ and $z = 0$ in (1) we get $x = \frac{85}{27} = 3.1481$

Putting $x = 3.1481$ and $z = 0$ in (2) we get

$$y = \frac{1}{15}[72 - 6 \times 3.1481] = 3.5408$$

Putting $x = 3.1481$ and $y = 3.5408$ in (3) we get

$$z = \frac{1}{54}[110 - 3.1481 - 3.5408] = 1.9132$$

Second iteration

Putting $y = 3.5408$ and $z = 1.9132$ in (1) we get

$$x = \frac{1}{27}[85 - 6 \times 3.5408 + 1.9132] = 2.4322$$

Putting $x = 2.4322$ and $z = 1.9132$ in (2) we get

$$y = \frac{1}{15}[72 - 6 \times 2.4322 - 2 \times 1.9132] = 3.572$$

Putting $x = 2.4322$ and $y = 3.572$ in (3) we get

$$z = \frac{1}{54}[110 - 2.4322 - 3.572] = 1.9258$$

Third iteration

Putting $y = 3.572$ and $z = 1.9258$ in (1) we get

$$x = \frac{1}{27}[85 - 6 \times 3.572 + 1.9258] = 2.4257$$

Putting $x = 2.4257$ and $z = 1.9258$ in (2) we get

$$y = \frac{1}{15}[72 - 6 \times 2.4257 - 2 \times 1.9258] = 3.5729$$

Putting $x = 2.4257$ and $y = 3.5729$ in (3) we get

$$z = \frac{1}{54}[110 - 2.4257 - 3.5729] = 1.926.$$

Fourth iteration

Putting y = 3.5729 and z = 1.926 in (1) we get

$$x = \frac{1}{27}[85 - 6 \times 3.5729 + 1.926] = 2.4255$$

Putting x = 2.4255 and z = 1.926 in (2) we get

$$y = \frac{1}{15}[72 - 6 \times 2.4255 - 2 \times 1.926] = 3.573$$

Putting x = 2.4255 and y = 3.573 in (3) we get

$$z = \frac{1}{54}[110 - 2.4255 - 3.573] = 1.926.$$

The values of x, y, z in the third and fourth iteration are almost equal.

$\therefore$ The roots of the system are x = 2.4255; y = 3.573 and z = 1.926.

Note: The above iterations can be simply carried out and shown in the following tabular form.

Iteration	Initial Value	1	2	3	4	5
x	-	3.1481	2.4322	2.4257	2.4255	2.4255
y	0	3.5408	3.5720	3.5729	3.5730	3.5730
z	0	1.9132	1.9258	1.9260	1.9260	1.9260

Problem 3: Solve the following system of equations using Gauss Seidel iteration method.

$10x + 2y + z = 9$ $\qquad$ $x + 10y - z = -22$ $\qquad$ $-2x + 3y + 10z = 22$.

Solution: Clearly the given system of equations is diagonally dominant. Hence it can be solved by Gauss-Seidel iteration method.

The given system of equations can be written as

$$x = \frac{1}{10}(9 - 2y - z) \qquad \text{------ (1)}$$

$$y = \frac{1}{10}(-22 - x + z) \qquad \text{------ (2)}$$

$$x = \frac{1}{10}(22 + 2x - 3y) \qquad \text{------ (3)}$$

First iteration

Putting y = 0 and z = 0 in (1) we get x = 0.9

Putting x = 0.9 and z = 0 in (2) we get

$$y = \frac{1}{10}(-22-0.9) = -2.29$$

Putting x = 0.9 and y = -2.29 in (3) we get

$$z = \frac{1}{10}(22+2\times 0.9-3(-2.29)) = 3.067$$

Second iteration

Putting y = -2.29 and z = 3.067 in (1) we get

$$x = \frac{1}{10}[9-2(-2.29)-3.067] = 1.0513$$

Putting x = 1.0513 and z = 3.067 in (2) we get

$$y = \frac{1}{10}(-22-1.0513+3.067) = -1.9984.$$

Putting x = 1.0513 and y = -1.9984 in (3) we get

$$z = \frac{1}{10}[22+2\times 1.0513-3(-1.9984)] = 3.0098.$$

Third iteration

Putting y = -1.9984 and z = 3.0098 in (1) we get

$$z = \frac{1}{10}[9-2(-1.9984)-3.0098] = 0.9987$$

Putting x = 0.9987 and z = 3.0098 in (2) we get

$$y = \frac{1}{10}[-22-0.9987+3.0098] = -1.9989$$

Putting x = 0.9987 and y = -1.9989 in (3) we get

$$z = \frac{1}{10}[22+2\times 0.9987-3(-1.9989)] = 2.9994$$

Fourth iteration

Putting y = -1.9989 and z = 2.9994 in (1) we get

$$x = \frac{1}{10}[9 - 2(-1.9989) - 2.9994] = 0.9998$$

Putting x = 0.9998 and z = 2.9994 in (2) we get

$$y = \frac{1}{10}[-22 - 0.9998 + 2.9994] = -2$$

Putting x = 0.9998 and y = -2 in (3) we get

$$z = \frac{1}{10}[22 + 2 \times 0.9998 - 3(-2)] = 3.$$

Proceeding like this in the next iteration we get

$x = 1$, $y = -2$ and $z = 3$.

$\therefore$ The roots are $x = 1$, $y = -2$ and $z = 3$.

Problem 4. Solve the following system of equations by

(i) Gauss-Seidel method (ii) Gauss-Jacobi method

$28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 17y + 4z = 35$.

Solution: The coefficient matrix of the system is A = $\begin{pmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{pmatrix}$. We find that A is not diagonally dominant. However

$$A \sim \begin{pmatrix} 28 & 4 & -1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{pmatrix} R_2 \leftrightarrow R_3;$$

which is diagonally dominant.

Hence the given equations become

$28x + 4y - z = 32$ ------ (1)

$2x + 17y + 4z = 35$ ------ (2)

$x + 3y + 10z = 24$ ------ (3)

From (1), (2) and (3) we have

$$x = \frac{1}{28}[32 - 4y + z] \qquad \text{------ (4)}$$

$$y = \frac{1}{17}[35 - 2x - 4z] \qquad \text{------ (5)}$$

$$z = \frac{1}{10}[24 - x - 3y] \qquad \text{------ (6)}$$

(i) **Gauss-Seidel Method**

First iteration

Put y = 0 and z = 0 in (4) we get $y = \frac{1}{28}[32] = 1.1429$

Put x = 1.1429; z = 0 in (5) we get $x = \frac{1}{17}[35 - 2.2858] = 1.9244$

Put x = 1.1429; y = 1.9244 in (6) then $z = \frac{1}{10}[24 - 1.1429 - 5.7732] = 1.7084$

After first iteration we have x = 1.1429, y = 1.9244, z = 1.7084

Second iteration

Put y = 1.9244 and z = 1.7084 in (4); $x = \frac{1}{28}[32 - 7.6976 + 1.7084] = 0.9290$

Put x = 0.9290; and z = 1.7084 in (5); $y = \frac{1}{17}[35 - 1.858 - 6.8336] = 1.5476$

Put x = 0.929; and y = 1.5476 in (6) $z = \frac{1}{10}[24 - 0.929 - 4.6428] = 1.8428$

After second iteration we have x = 0.929, y = 1.5476, z = 1.8428.

Third iteration

Put y = 1.5476 and z = 1.8428 in (4); $x = \frac{1}{28}[32 - 6.1904 + 1.8428] = 0.9876$

Put x = 0.9876 and z = 1.8428 in (5); $y = \frac{1}{17}[35 - 1.9752 - 7.3712] = 1.5090$

Put x = 0.9876 and y = 1.5090 in (6); $z = \frac{1}{10}[24 - 0.9876 - 4.527] = 1.8485$

After third iteration we have x = 0.9876, y = 1.9426, z = 1.7155.

Fourth iteration

Put y = 1.5090 and z = 1.8485 in (4); $x = \frac{1}{28}[32 - 6.036 + 1.885] = 0.9933$

Put x = 0.9933 and z = 1.8485 in (5); $y = \frac{1}{17}[35 - 1.9866 - 7.394] = 1.5070$

Put x = 0.9933 and y = 1.507 in (6); $z = \frac{1}{10}[24 - 0.9266 - 4.521] = 1.8486$

After fourth iteration we have x = 0.9933, y = 1.5070, z = 1.8486.

Fifth iteration

Put y = 1.5070 and z = 1.8486 in (4); $x = \frac{1}{28}[32 - 6.028 + 1.8486] = 0.9936$

Put x = 0.9936 and z = 1.8486 in (5); $y = \frac{1}{17}[35 - 1.9872 - 7.3944] = 1.5070$

Put x = 0.9936 and y = 1.5070 in (6); $z = \frac{1}{10}[24 - 0.9936 - 4.521] = 1.8485$

After fifth iteration we have x = 0.9936, y = 1.5070, z = 1.8485.

Fourth and fifth iterations give almost the same values.

Hence x = 0.9936, y = 1.5070, z = 1.8485.

(ii) Gauss-Jacobi Method

Gauss Jacobi iteration formula is

$$x^{(r+1)} = \frac{1}{a_1}\left[d_1 - b_1 y^{(r)} - c_1 z_r^{(r)}\right]$$

$$y^{(r+1)} = \frac{1}{b_2}\left[d_2 - a_2 x^{(r)} - c_2 y^{(r)}\right]$$

$$y^{(r+1)} = \frac{1}{c_3}\left[d_3 - a_3 x^{(r)} - c_3 y^{(r)}\right]$$

First iteration. Let the initial values be $x = y = z = 0$.

Put $x = y = z = 0$ in (4) we get

$$x_1 = \frac{32}{28} = 1.429\,; \qquad y_1 = \frac{35}{17} = 2.0588\,; \qquad z_1 = \frac{24}{10} = 2.4$$

Second iteration

$$x_2 = \frac{1}{28}[32 - 4y_1 + z_1] = \frac{1}{28}[32 - 8.2352 + 2.4] = 0.9345$$

$$y_2 = \frac{1}{17}[35 - 2x_1 - 4z_1] = \frac{1}{17}[32 - 2.2858 - 9.6] = 1.3597$$

$$z_2 = \frac{1}{10}[24 - x_1 - 3y_1] = \frac{1}{10}[24 - 1.1429 - 6.1764] = 1.16681$$

Third iteration

$$x_3 = \frac{1}{28}[32 - 4y_2 + z_2] = \frac{1}{28}[32 - 5.4388 + 1.6681] = 1.0082$$

$$y_3 = \frac{1}{17}[35 - 2x_2 - 4z_2] = \frac{1}{17}[35 - 1.869 - 6.6724] = 1.5564$$

$$z_3 = \frac{1}{10}[24 - x_2 - 3y_2] = \frac{1}{10}[24 - 0.9345 - 4.0791] = 1.8986$$

Fourth iteration

$$x_4 = \frac{1}{28}[32 - 4y_3 + z_3] = \frac{1}{28}[32 - 6.2256 + 1.8986] = 0.9883$$

$$y_4 = \frac{1}{17}[35 - 2x_3 - 4z_3] = \frac{1}{17}[35 - 2.0164 - 7.5944] = 1.4935$$

$$z_4 = \frac{1}{10}[24 - x_3 - 3y_3] = \frac{1}{10}[24 - 1.0082 - 4.6692] = 1.8323$$

Fifth iteration

$$x_5 = \frac{1}{28}[32 - 4y_4 + z_4] = \frac{1}{28}[32 - 5.974 + 1.8323] = 0.9949$$

$$y_5 = \frac{1}{17}[35 - 2x_4 - 4z_4] = \frac{1}{17}[32 - 1.9766 - 7.3292] = 1.5114$$

$$z_5 = \frac{1}{10}[24 - x_4 - 3y_4] = \frac{1}{10}[24 - 0.9883 - 4.4805] = 1.8531$$

Sixth iteration

$$x_6 = \frac{1}{28}[32 - 4y_5 + z_5] = \frac{1}{28}[32 - 6.0456 + 1.8531] = 0.9931$$

$$y_6 = \frac{1}{17}[35 - 2x_5 - 4z_5] = \frac{1}{17}[35 - 1.9898 - 7.4124] = 1.5058$$

$$z_6 = \frac{1}{10}[24 - x_5 - 3y_5] = \frac{1}{10}[24 - 0.9949 - 4.5342] = 1.8471$$

Seventh iteration

$$x_7 = \frac{1}{28}[32 - 4y_6 + z_6] = \frac{1}{28}[32 - 6.0232 + 1.8471] = 0.9937$$

$$y_7 = \frac{1}{17}[35 - 2x_6 - 4z_6] = \frac{1}{17}[35 - 1.9862 - 7.3884] = 1.5074$$

$$z_7 = \frac{1}{10}[24 - x_6 - 3y_6] = \frac{1}{10}[24 - 0.9931 - 4.5174] = 1.8490$$

Eighth iteration

$$x_8 = \frac{1}{28}[32 - 4y_7 + z_7] = \frac{1}{28}[32 - 6.0296 + 1.8490] = 0.9936$$

$$y_8 = \frac{1}{17}[35 - 2x_7 - 4z_7] = \frac{1}{17}[35 - 1.9874 - 7.396] = 1.5069$$

$$z_8 = \frac{1}{10}[24 - x_7 - 3y_7] = \frac{1}{10}[24 - 0.9937 - 4.5222] = 1.8484$$

The values in the seventh and eighth iterations are close to each other.

Hence the solution is given by $x = 0.9936$, $y = 1.5069$, $z = 1.8484$

Remark. We observe that in the above problem Gauss-Seidel method gives the answer in five iterations whereas Gauss Jacobi method gives more or less the same answer only after eight iterations.

Exercises

Solve the following system of equations by Gauss Seidel iteration Method

1. 8x - y + z = 18 ; 2x + 5y - 2z = 3; x + y -3z = -6
2. 8x + y + z = 8 ; 2x + 4y + z = 4; x + 3y + 3z =5

4.4 Relaxation Method

We describe this method only for a system of *three equations* in *three unknowns* given by

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

We define the *residuals*

$$\begin{aligned} R_x &= a_1x+b_1y+c_1z-d_1 \\ R_y &= a_2x+b_2y+c_2z-d_2 \quad \ldots\ldots(1) \\ R_z &= a_3x+b_3y+c_3z-d_3 \end{aligned}$$

We observe that for actual solution of the system the residuals become zero.

The relaxation method consists of reducing the values of the residuals as close to zero as possible by modifying the values of the variables at each stage. For this purpose we give an *operation table* from which we can know the changes in R_x, R_y, R_z corresponding to any change in the values of the variables.

The operation table is given by

Δx	Δy	Δz		ΔR_x	ΔR_y	ΔR_z
1	0	0		$-a_1$	$-a_2$	$-a_3$
0	1	0		$-b_1$	$-b_2$	$-b_3$
0	0	1		$-c_1$	$-c_2$	$-c_3$

We note from the equations (1) that if x is increased by 1 (keeping y and z constant) R_x, R_y, R_z decrease by a_1, a_2, a_3 respectively. This is shown in the above table along with the effects on the residuals when y and z are given unit increments.

At each step the *numerically largest* residual is reduced to almost zero. To reduce a particular residual the value of the corresponding variable is changed. When all the residuals are reduced to *almost zero* the increments in x, y, z are added separately to give the desired solution

$$x = \Sigma \Delta x; \qquad y = \Sigma \Delta y; \qquad z = \Sigma \Delta z.$$

Problem 1. Solve the following equations using relaxation method.

$5x - y - z = 3$; $\quad -x + 10y - 2z = 7$; $\quad -x - y + 10z = 8$

Solution: We note that the coefficient matrix $A = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 10 & -2 \\ -1 & -1 & 10 \end{pmatrix}$ is diagonally dominant and hence relaxation method can be applied. The residuals are given by

$$R_x = 5x - y - z - 3$$

$$R_y = -x + 10y - 2z - 7$$

$$R_z = -x - y + 10z - 8$$

Operation Table

Increment			ΔR_x	ΔR_y	ΔR_z	Explanation
Δx	Δy	Δz				
1	0	0	5	-1	-1	x is initialized by 1, $y = 0 = z$
0	1	0	-1	10	-1	y is initialized by 1, $x = 0 = z$
0	0	1	-1	-2	10	z is initialized by 1, $x = 0 = y$

Relaxation Table

Increment			R_x	R_y	R_z	Explanation
Δx	Δy	Δz				
0	0	0	-3	-7	-8	The numerically largest residual is -8 Increment in z is given by $\Delta z = 1$
0	0	1	-4	-9	2	The numerically largest residual is -9 Increment in y is given by $\Delta y = 1$
0	1	0	-5	1	1	The numerically largest residual is -5 Increment in x is given by $\Delta x = 1$
1	0	0	0	0	0	All the residuals have been reduced to zero

In the final iteration the residuals are zero.

$\therefore \Delta\Sigma x = 1$; $\Delta\Sigma y = 1$ and $\Delta\Sigma z = 1$ gives the solution as

$x = 1;$ $y=1;$ $x = 1.$

Problem 2. Solve the following equations using relaxation method.

$9x - y + 2z = 9;$ $x + 10y - 2z = 15;$ $2x - 2y - 13z = -17$

Solution. We note that the given system is a diagonal system. Hence relaxation method can be used to solve. The residuals are

$$R_x = 9x - y + 2z - 9$$
$$R_y = x + 10y + 2z - 15$$
$$R_z = 2x - 2y - 13z + 17$$

Operation Table

Increment			ΔR_x	ΔR_y	ΔR_z	Explanation
Δx	Δy	Δz				
1	0	0	9	1	2	x is initialized as 1, when $y = 0 = z$
0	1	0	– 1	10	– 2	y is initialized as 1, when $x = 0 = z$
0	0	1	2	2	– 13	z is initialized as 1, when $x = 0 = y$

Relaxation Table

Increment			ΔR_x	ΔR_y	ΔR_z	Explanation
Δx	Δy	Δz				
0	0	0	– 9	– 15	17	Numerically largest residual is $R_z = 17$ Increment in z is given by $\Delta z = 1$
0	0	1	– 7	– 17	4	Numerically largest residual is $R_y = -17$ Increment in y is given by $\Delta y = 2$
0	2	0	– 9	3	0	Numerically largest residual is $R_x = -9$ Increment in x is given by $\Delta x = 1$
1	0	0	0	4	2	Numerically largest residual is $R_y = 4$ Increment in y is given by $\Delta y = -0.4$
0	– 0.4	0	0.4	0	2.8	Numerically largest residual is $R_z = 2.8$ Increment in z is given by $\Delta z = 0.2$
0	0	0.2	0.8	– 0.4	0.2	Numerically largest residual is $R_x = 0.8$ Increment in x is given by $\Delta x = -0.1$

– 0.1	0	0	– 0.1	– 0.5	0	Numerically largest residual is $R_y = -\ 0.5$ Increment in y is given by $\Delta y = 0.02$
0	0.05	0	– 0.15	0	– 0.10	Numerically largest residual is $R_x = -\ 0.15$ Increment in x is given by $\Delta x = 0.02$
0.02	0	0	0.030	0.020	– 0.06	Numerically largest residual is $R_z = -\ 0.06$ Increment in z is given by $\Delta z = -\ 0.005$
0	0	– 0.005	– 0.20	0.030	0.005	Numerically largest residual is $R_y = 0.030$ Increment in y is given by $\Delta y = -0.003$
0	– .003	– 0.005	0.023	0	0.011	Numerically largest residual is $R_x = 0.023$ Increment in x is given by $\Delta x = -0.003$
– 0.003	0	0	– .004	– .003	0.005	The residuals are reduced almost to zero. Iteration stops

Now $\Sigma \Delta x = 1 + (-0.1) + 0.02 + (-0.003) = 0.917$

$\Sigma \Delta y = 2 + (-0.4) + 0.05 + (-0.003) = 1.647$

$\Sigma \Delta z = 1 + 0.02 + (-0.005) = 1.195$

Hence $x = 0.917$; $y = 1.647$; $z = 1.195$

Exercises

Solve the following system of equations by Relaxation Method

1. 10x - 2y + z = 12 ; x + 9y - z = 10; 2x - y + 11z = 20
2. 9x - 2y + z = 50 ; x +5 y - 3z = 18; -2x + 2y + 7z =19

4.5 Newton Raphson Method For Simultaneous Equations

Consider the equations $f(x, y)$ 0, $g(x, y) = 0$. … (1)

Let (x_0, y_0) be an initial approximate solution of (1)

Let $x_1 = x_0 + h$ and $y_1 = y_0 + k$ be the next approximation.

Expending f and g by Taylor's theorem for a function of two variables around the point (x_1, y_1), we have

$$f(x_1, y_1) = f(x_0 + h, y_0 + k) = f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} + k\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)}$$

$$f(x_1, y_1) = g(x_0 + h, y_0 + k) = g(x_0, y_0) + h\left(\frac{\partial g}{\partial x}\right)_{(x_0, y_0)} + k\left(\frac{\partial g}{\partial x}\right)_{(x_0, y_0)}$$

(omitting higher powers of h and k)

If (x_1, y_1) is a solution of (1), then $f(x_1, y_1) = 0$ and $g(x_1, y_1) = 0$.

$$\text{Hence } f(x_0, y_0) + hf_x(x_0, y_0) + kf_y(x_0, y_0) = 0$$

$$\text{and } g(x_0, y_0) + hg_x(x_0, y_0) + kg_y(x_0, y_0) = 0$$

$$\therefore hf_x(x_0, y_0) + kf_y(x_0, y_0) = -f(x_0, y_0) \qquad \ldots (2)$$

$$hg_x(x_0, y_0) + kg_y(x_0, y_0) = -g(x_0, y_0) \qquad \ldots (3)$$

If the Jacobian $J = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \neq 0$, then the equations (2) and (3) provide a unique solution for h and k. Now $x_1 = x_0 + h$ and $y_1 = y_0 + k$ give a new approximation to the solution.

By repeating this process we obtain the required solution to the desired accuracy.

Problem 1 Solve the equations $x = x^2 + y^2$, $y = x^2 - y^2$ using Newton Raphson method with the approximation (0.8, 0.4)

Solution. Let $f(x, y) = x - x^2 - y^2$; $g(x, y) = y - x^2 + y^2$

$$\therefore f_x = \frac{\partial f}{\partial x} = 1 - 2x; \qquad f_y = \frac{\partial f}{\partial x} = -2y$$

$$g_x = \frac{\partial g}{\partial x} = -2x; \qquad g_y = \frac{\partial g}{\partial x} = 1 + 2y$$

Here $x_0 = 0.8$ and $y_0 = 0.4$

$$\therefore f(x_0, y_0) = x_0 - x_0^2 + y_0^2 = 0$$

$$\text{and } g(x_0, y_0) = y_0 - x_0^2 + y_0^2 = -0.08$$

$$\text{Also } f_x(x_0, y_0) = 1 - 2x_0 = -0.6$$

$$f_y(x_0, y_0) = -2y_0 = -0.8$$

$$g_x(x_0, y_0) = -2x_0 = -1.6$$

$$\text{and } g_y(x_0, y_0) = 1 + 2y_0 = 1.8$$

The Newton Raphson's equations are

$$f(x_0,y_0)+hf_x(x_0,y_0)+kf_y(x_0,y_0) \quad = 0 \qquad \ldots (1)$$

$$g(x_0,y_0)+hf_x(x_0,y_0)+kg_y(x_0,y_0) \quad = 0 \qquad \ldots (2)$$

i.e. 0.6 h + 0.8 k = 0

– 1.6 h + 1.8 k = 0.08

Solving these two equations we get

h = – 0.027 and k = 0.02

The next approximation to the solution is

$$x_1 = x_0 + h = 0.773 \text{ and } y_1 = y_0 + k = 0.42$$

Now $f(x_1,y_1) = -0.009$; $g(x_1,y_1) = -0.0011$; $f_x(x_1,y_1) = -0.546$; $f_y(x_1,y_1) = -0.84$; $g_x(x_1,y_1) = -1.546$; and $g_y(x_1,y_1) = 1.84$

Using these values in (1) and (2) we have

$$-0.546h - 0.84k = 0.0009$$

$$-1.546h + 1.84k = 0.0011$$

Solving for h and k we get h = – 0.0011 and k = – 0.0004

$$\therefore \; x_2 = x_1 + h = 0.7719$$

$$y_2 = y_1 + h = 0.4196$$

Thus x = 0.7719 and y = 0.4196 is an approximate solution to the given system.

Problem 2: Use Newton-Raphson method to solve the equations $x^2 - y^2 = 4$. $x^2 + y^2 = 16$ with $x_0 = y_0 = 2.828$.

Solution: Let $f = x^2 - y^2 - 4$ and $g = x^2 + y^2 - 16$

$$\therefore \quad f_x = 2x, \; f_y = -2y$$

$$g_x = 2x, \; g_y = 2y$$

Also $f(x_0,y_0) = x_0^2 - y_0^2 - 4 = 0$; $g(x_0,y_0) = x_0^2 + y_0^2 - 16 = 0$

$f_x(x_0, y_0) = 2x_0 = 5.656$

$f_y(x_0, y_0) = -2y_0 = -5.656$

$g_x(x_0, y_0) = 2x_0 = 5.656$

$g_y(x_0, y_0) = 2y_0 = 5.656$.

The Newton Raphson equations are

$$f(x_0, y_0) + hf_x(x_0, y_0) + kf_y(x_0, y_0) = 0 \quad \ldots (1)$$

$$g(x_0, y_0) + hg_x(x_0, y_0) + kg_y(x_0, y_0) = 0 \quad \ldots (2)$$

i.e. $h - k = 0.707$

and $h + k = 0$

$\therefore h = 0.354$ and $k = -0.354$.

The next approximation to the solution is

$x_1 = x_0 + h = 3.182$

$y_1 = y_0 + k = 2.474$

Now $f(x_1, y_1) = x_1^2 - y_1^2 - 4 = 0.004448$

$g(x_1, y_1) = x_1^2 + y_1^2 - 16 = 0.2458$

$f_x(x_1, y_1) = 2x_1 = 6.364$

$f_y(x_1, y_1) = -2y_1 = -4.948$

$g_x(x_1, y_1) = 2x_1 = 6.364$

$g_y(x_1, y_1) = -2y_1 = 4.948$

Replacing (x_0, y_0) by (x_1, y_1) in (1) and (2) we get the Newton Raphson equations for the second approximation as

$$6.364h - 4.948k = -0.0044$$

$$6.364h + 4.9484k = -0.2458$$

Solving these two equation we get

h = – 0.01966 and k = – 0.0244

$$\therefore\ x_2 = x_1 + h = 3.162$$

$$y_2 = y_1 + k = 2.45$$

Thus x = 3.162 and y = 2.45 is an approximate solution to the given system.

Problem 3 Find a root of the system of non-linear equations by Newton-Raphson method: $x^2 + y = 11$, $y^2 + x = 7$, with $x_0 = 3.5$ and $y_0 = -1.8$.

Solution: Let $f = x^2 + y - 11$ and $g = y^2 + x - 7$.

$$\therefore\ f(x_0, y_0) = x_0^2 + y_0 - 11 = -0.55$$

$$g(x_0, y_0) = y_0^2 + x_0 - 7 = -0.26$$

$$f_x(x_0, y_0) = 2x_0 = 7$$

$$f_y(x_0, y_0) = 1$$

$$g_x(x_0, y_0) = 1$$

$$g_y(x_0, y_0) = 2y_0 = -3.6$$

Newton-Raphson equation are

$$f(x_0, y_0) + h\, f_x(x_0, y_0) + k\, f_y(x_0, y_0) = 0 \qquad \text{....(1)}$$

$$g(x_0, y_0) + h\, g_x(x_0, y_0) + k\, j_y(x_0, y_0) = 0 \qquad \text{.....(2)}$$

i.e. $7h + k = 0.55$

$$h - 3.6k = 0.26$$

Solving these equation we get h = 0.0855 and k = - 0.0485

∴ The next approximation to the solution is

$$x_1 = x_0 + h = 3.5855,\ y_1 = y_0 + k = -1.8485$$

Now $f(x_1, y_1) = x_1^2 + y_1 - 11 = 0.0073$

$$g(x_1, y_1) = y_1^2 + x_1 - 7 = 0.0025$$

$$f_x(x_1, y_1) = 2x_1 = 7.171$$

$$f_y(x_1, y_1) = 1$$

$$g_x(x_1, y_1) = 1$$

$$g_y(x_1, y_1) = 2y_1 = -3.697$$

Replacing (x_0, y_0) by (x_1, y_1) in (1) and (2) we get the Newton-Raphson equation for the next approximation as

$$7.171\,h + k = -0.0073$$

$$h - 3.697\,k = -0.0025$$

Solving these two equation we get

$$h = -0.00106 \text{ and } k = 0.00039$$

i.e. $x_2 = x_1 + h = 3.5844$

$$y_2 = y_1 + k = -1.8481$$

Thus x = 3.5844 and y = -1.8481 is an approximate solution to the given system.

Problem 4 Solve the system of equations sin $xy + x - y = 0$; y cos $xy + 1 = 0$ with. $x_0 = 1$ and $y_0 = 2$ by Newton Raphson method.

Solution: Let $f(x, y) = \sin xy + x - y$; $g(x, y) = y \cos xy + 1$

$$\therefore \quad f_x = y \cos xy + 1$$

$f_y = x \cos xy - 1$

$g_x = -y^2 \sin xy$

$g_y = -xy \sin xy$

With $x_0 = 1$ and $y_0 = 2$ we have

$f(x_0, y_0) = \sin x_0 y_0 + x_0 - y_0 = 0.9092 + 1 - 2 = -0.0907$

$g(x_0, y_0) = y_0 \cos x_0 y_0 + 1 = -0.8323 + 1 = 0.1677$

$f_x(x_0, y_0) = 0.1677$

$f_y(x_0, y_0) = -1.4161$

$g_x(x_0, y_0) = -3.6372$

$g_y(x_0, y_0) = -1.8186$

The Newton Raphson equations are

$f(x_0, y_0) + h\, f_x(x_0, y_0) + k\, f_y(x_0, y_0) = 0$ ..... (1)

$g(x_0, y_0) + h\, g_x(x_0, y_0) + k\, g_y(x_0, y_0) = 0$ ..... (2)

i.e. $0.1677\, h - 1.4261\, k = 0.0907$

$3.6372\, h + 1.8186\, k = 0.1677$

Solving these two equation we get

$h = 0.0739$ and $k = -0.0553$

$\therefore x_1 = x_0 + h = 1.0739$ and $y_1 = y_0 + k = 1.9447$

Now $f(x_1, y_1) = \sin x_1 y_1 + x_1 - y_1 = 0.0018$

$g(x_1, y_1) = y_1 \cos x_1 y_1 + 1 = 0.0378$

$f_x(x_1, y_1) = y_1 \cos x_1 y_1 + 1 = 0.0378$

$f_y(x_1, y_1) = x_1 \cos x_1 y_1 - 1 =$ -1.5314

$g_x(x_1, y_1) = -y_1^2 \sin x_1 y_1 =$ -3.2865

$g_y(x_1, y_1) = -x_1 y_1 \sin x_1 y_1 =$ - 1.8148

Replacing (x_0, y_0) by (x_1, y_1) in (1) and (2) we get the Newton-Raphson equation for the second approximation as

$$0.03784h - 1.5314k = -0.0018$$

$$3.2865\,h + 1.8148k = 0.0378$$

Solving these who equations we get h = 0.0089 and k = 0.0014.

i.e., $x_2 = x_1 + h = 1.0828$ and $y_2 = y_1 + k = 1.9461$

Thus x = 1.0828 and y = 1.9461 is an approximate solution for the given system.

Exercises

Solve the following system of simultaneous equations by Newton Raphson method

1. $xy = 1$; $x^2 + y^2 = 4$ with $x_0 = 1.8$ and $y_0 = 0.5$
2. $xy = x + y$; $y^2 + x^2 = 1$ with $x_0 = 0.5$, $y_0 = -1$

Numerical Solution of Partial Differential Equations

Unit V

5.1 Classification of partial differential equations of second order

Definition: The general second order linear partial differential equation in two independent variables is of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

which can be written as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A,B,C,D,E,F,G are functions of x and y

The above equation is said to be **elliptic** or **parabolic** or **hyperbolic** at a point (x, y) in the plane according as $B^2 - 4AC < 0$ or $B^2 - 4AC = 0$ or $B^2 - 4AC > 0$.

Note: It is possible for a second order partial differential equation to be elliptic in one region, parabolic in another and hyperbolic in some other region.

For example consider

$$xu_{xx} + u_{yy} = 0 \qquad \ldots (1)$$

Here $A = x$; $B = 0$; $C = 1$ so that $B^2 - 4AC = -4x$.

The equation (1) is elliptic if $B^2 - 4AC < 0$ (i.e) $-4x < 0$ (i.e) if $x > 0$.

Similarly (1) is parabolic if $x = 0$ and hyperbolic if $x < 0$.

Example 1: The Laplace equation $u_{xx} + u_{yy} = 0$ and the Poisson's equation $u_{xx} + u_{yy} = f(x,y)$ are elliptic.

2. The one-dimensional heat equation $u_t = \alpha^2 u_{xx}$ is parabolic.

3. The one-dimensional wave equation $u_{tt} = \alpha^2 u_{xx}$ is hyperbolic.

Problem 1: Classify the equations

(i) $u_{xx} + 2u_{xy} + u_{yy} = 0$

(ii) $x^2 f_{xx} + (1 - y^2) f_{yy} = 0$

(iii) $u_{xx} + 4u_{xy} + (x^2 + 4y^2) u_{yy} = \sin xy$

(iv) $(1 + x^2) u_{xx} + (5 + 2x^2) u_{xt} + (4 + x^2) u_{tt} = 0$

Solution (i) Comparing the given equation with the general second order linear partial differential equation we have A = 1; B = 2; C = 1.

Now $B^2 - 4AC = 4 - 4 = 0$

$\therefore$ The given equation is parabolic.

(ii) Here $A = x^2$; B = 0; $C = 1 - y^2$

Now $B^2 - 4AC = 0 - 4x^2(1 - y^2) = 4x^2(y^2 - 1)$

We note that $x^2 > 0$ for all x except x = 0 and $y^2 - 1 < 0$ for all y such that $-1 < y < 1$

$\therefore B^2 - 4AC < 0$ for all $x \neq 0$ and $-1 < y < 1$

$\therefore$ The equation is elliptic in the region given by $x \neq 0$ and $-1 < y < 1$.

Similarly the equation is hyperbolic in the region given by $x \neq 0$ and $y < -1$ or $y > 1$.

$B^2 - 4AC = 4x^2(y^2 - 1) = 0$ if $x = 0$ or $y^2 = 1$

$\therefore$ The equation is parabolic if $x = 0$ or $y = \pm 1$.

(iii) $u_{xx} + 4u_{xy} + (x^2 + 4y^2) u_{yy} = \sin xy$.

Here $A = 1$; $B = 4$ and $C = x^2 + 4y^2$

Now $B^2 - 4AC = 16 - 4(x^2 + 4y^2) = 4[4 - x^2 - 4y^2]$

The equation is elliptic if $4 - x^2 - 4y^2 < 0$

i.e. if $x^2 + 4y^2 > 4$

i.e. if $\frac{x^2}{4} + \frac{y^2}{1} > 1$

$\therefore$ The equation is elliptic in the region outside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

It is Hyperbolic inside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

It is parabolic on the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

(iv) $\left(1+x^2\right)u_{xx}+\left(5+2x^2\right)u_{xt}+\left(4+x^2\right)u_{tt}=0$

Here $A=1+x^2$; $B=5+2x^2$; $C=4+x^2$

$$\begin{aligned} \text{Now } B^2-4AC &= \left(5+2x^2\right)^2-4\left(1+x^2\right)\left(4+x^2\right) \\ &= 25+4x^4+20x^2-4\left(4+5x^2+x^4\right) \\ &= 25+4x^4+20x^2-16-20x^2-4x^4 \\ &= 9. \text{ (positive)} \end{aligned}$$

Hence the equation is hyperbolic.

Exercises

Classify the following partial differential equations

1. $f_{xx} - 2f_{xy} = 0$
2. $u_{xx} - 2u_{xy} - 8u_{yy} = 0$

5.2 Finite Difference Approximations to Derivatives

We divide the *xy*-plane into a network of rectangles of sides *h* and *k* by drawing the lines $x = ih$ and $y = jk$, $i, j = 0, 1, 2, \ldots$. The points of intersections of these family of lines are called mesh points or grid points or lattice points.

Now let $u(x, y)$ be a function of two independent variables *x* and *y*. By definition of partial derivative we have

$$\frac{\partial u}{\partial x} = u_x = \lim_{x\to 0}\frac{u(x+h,y)-u(x,y)}{h}$$

$$\therefore \quad u_x = \frac{u(x+h,y)-u(x,y)}{h}+O(h)$$

Also $$u_x = \frac{u(x,y)-u(x-h,y)}{h}+O(h)$$ and

$$u_x = \frac{u(x+h,y)-u(x-h,y)}{2h} + O(h^2)$$

Let $u(x,y) = u(ih, jk) = u_{i,j}$.

Then the above equations take the form

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h)$$

$$u_x = \frac{u_{i,j} - u_{i-1,j}}{h} + O(h)$$

$$u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2)$$

Similarly for the derivative $\frac{\partial u}{\partial y} = u_y$ we have

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k)$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k)$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2).$$

Similarly for the second order partial derivative u_{xx}, u_{yy} the finite difference approximation are given by

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2)$$

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2)$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations we get the finite difference analogue of the partial differential equation.

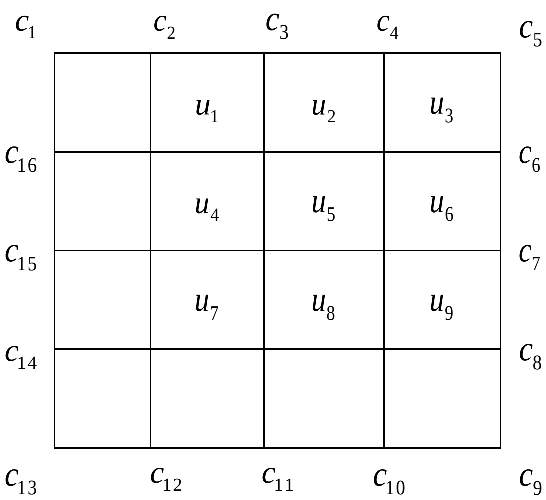
5.3 Laplace Equation

A numerical method for solving the two dimensional Laplace equation.

$$u_{xx} + u_{yy} = 0 \qquad \text{.... (1)}$$

Consider a rectangular region R for which $u(x,y)$ is known at the boundary. For simplicity we take R to be a square region and divide it into small squares of side h as shown in the Figure , where $c_1, c_2, ..., c_{16}$ are boundary values

Numerical Solution of Partial Differential Equations.



Figure

We now replace the partial derivatives in (1) by the difference approximations

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$$

$\therefore$ (1) becomes,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = 0$$

$$\therefore\ u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right] \qquad \text{... (2)}$$

(2) is called the standard five point formula (SFPF)

Since Laplace equation remains invariant when the coordinates are rotated through 45^0 we can also use the following formula instead of (2)

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right] \qquad \text{... (3)}$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighboring diagonal mesh points. (3) is called the diagonal five point formula (DFPF).

1. **Gauss Jocobi method**

Let $u_{i,j}^{(n)}$ denote the n[th] iterative value of $u_{i,j}$ then an iterative procedure to solve (2) is given by

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right] \text{ for the interior mesh points.}$$

2. **Gauss Seidel method**

The iterative procedure to solve (2) is given by

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right]$$

Here we use the latest available iterative values and hence the rate of convergence will be twice as fast as the Jacobi method.

Obtaining the solution by using Gauss-Seidel iteration method is known as

Leibmann's iterative process.

Problem 1. By iteration method solve the elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

over the square region of side 4 satisfying the boundary conditions

a) $u(0, y) = 0$ for $0 \le y \le 4$

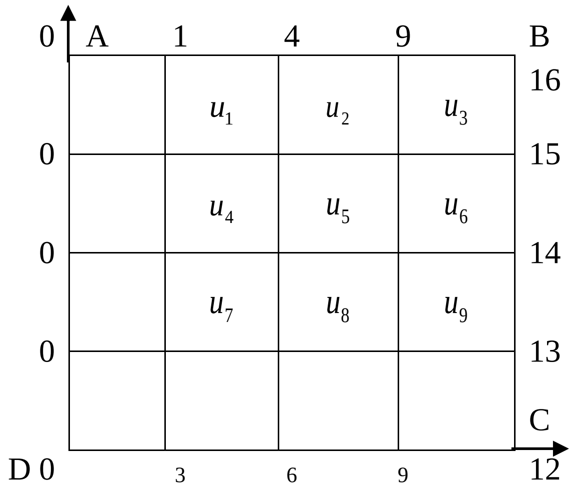
b) $u(4, y) = 12+y$ for $0 \le y \le 4$

c) $u(x, 0) = 3x$ for $0 \le x \le 4$

d) $u(x, 4) = x^2$ for $0 \le x \le 4$

By dividing the square into 16 square meshes of side 1 and always correcting the computed values to two places of decimals obtain the values of u at 9 interior pivotal points.

Solution: The region of $u(x,y)$ with the given boundary conditions are shown in Figure.



Figure

The boundary condition $u\ (0, y) = 0$ for $0 \leq y \leq 4$ gives all boundary values on the line *DA*, $x = 0$(y-axis).

The boundary condition $u(4, y) = 12+y$ for $0 \leq y \leq 4$ gives the four boundary values 13, 14, 15, 16 on the line CB, whose equation is $x = 4$.

The boundary condition $u\ (x, 0) = 3x$ for $0 \leq x \leq 4$ gives the four boundary values 3, 6, 9, 12 on the line DC, $y = 0$ (x-axis).

The boundary condition $u(x, 4) = x^2$ for $0 \leq x \leq 4$ gives the four boundary values 0, 1, 4, 9 on the line *AB*, whose equation is $y = 4$.

Let the values of u at the 9 interior grid points be $u_1, u_2, u_3, \ldots, u_9.$

We use Liebmann's iteration method to find the values of $u_1, u_2, u_3, \ldots, u_9.$

Step 1. To find the 9 rough values of $u_1, u_2, u_3, \ldots, u_9.$

$$u_5 = \frac{1}{4}(4+6+0+14) = 6 \qquad \text{(SFPF)}$$

$$u_1 = \frac{1}{4}(0+6+4+0) = 2.5 \qquad \text{(DFPF)}$$

$$u_3 = \frac{1}{4}(16+6+4+14) = 10 \qquad \text{(DFPF)}$$

$$u_7 = \frac{1}{4}(6+0+0+6) = 3 \qquad \text{(DFPF)}$$

$$u_9 = \frac{1}{4}(14+6+6+12) = 9.5 \quad \text{(DFPF)}$$

Now we find the other four values using SFPF

$$u_2 = \frac{1}{4}(4+6+2.5+10) = 5.63$$

$$u_4 = \frac{1}{4}(2.5+3+0+6) = 2.88$$

$$u_6 = \frac{1}{4}(10+9.5+6+14) = 9.88$$

$$u_8 = \frac{1}{4}(6+6+3+9.5) = 6.13$$

Step 2. First iteration. In all further calculations we use SFPF and the latest available values.

$$u_1 = \frac{1}{4}(1+2.88+0+5.63) = 2.38$$

$$u_2 = \frac{1}{4}(4+6+2.38+10) = 5.60$$

$$u_3 = \frac{1}{4}(9+9.88+5.60+15) = 9.87$$

$$u_4 = \frac{1}{4}(2.38+3+0+6) = 2.85$$

$$u_5 = \frac{1}{4}(5.6+6.13+2.85+9.88) = 6.12$$

$$u_6 = \frac{1}{4}(9.87+9.5+6.12+14) = 9.87$$

$$u_7 = \frac{1}{4}(2.85+3+0+6.13) = 3.00$$

$$u_8 = \frac{1}{4}(6.12+6+3.00+9.50) = 6.16$$

$$u_9 = \frac{1}{4}(9.87+9+6.16+13) = 9.51$$

Second iteration

$$u_1 = \frac{1}{4}(1 + 2.85 + 0 + 5.6) = 2.36$$

$$u_2 = \frac{1}{4}(4 + 6.12 + 2.36 + 9.87) = 5.59$$

$$u_3 = \frac{1}{4}(9 + 9.87 + 5.59 + 15) = 9.87$$

$$u_4 = \frac{1}{4}(2.36 + 3.00 + 0 + 6.12) = 2.87$$

$$u_5 = \frac{1}{4}(5.59 + 6.16 + 2.87 + 9.87) = 6.12$$

$$u_6 = \frac{1}{4}(9.87 + 9.51 + 6.12 + 14) = 9.88$$

$$u_7 = \frac{1}{4}(2.87 + 3 + 0 + 6.16) = 3.01$$

$$u_8 = \frac{1}{4}(6.12 + 6 + 3.01 + 9.51) = 6.16$$

$$u_9 = \frac{1}{4}(9.88 + 9 + 6.16 + 13) = 9.51$$

Third iteration

$$u_1 = \frac{1}{4}(1 + 2.87 + 0 + 5.59) = 2.37$$

$$u_2 = \frac{1}{4}(4 + 6.12 + 2.37 + 9.87) = 5.59$$

$$u_3 = \frac{1}{4}(9 + 9.88 + 5.59 + 15) = 9.87$$

$$u_4 = \frac{1}{4}(2.37 + 3.01 + 0 + 6.12) = 2.88$$

$$u_5 = \frac{1}{4}(5.59 + 6.16 + 2.88 + 9.88) = 6.13$$

$$u_6 = \frac{1}{4}(9.87 + 9.51 + 6.13 + 14) = 9.88$$

$$u_7 = \frac{1}{4}(2.88 + 3 + 0 + 6.16) = 3.01$$

$$u_8 = \frac{1}{4}(6.13 + 6 + 3.01 + 9.51) = 6.16$$

$$u_9 = \frac{1}{4}(9.88 + 9 + 6.16 + 13) = 9.51$$

Fourth iteration

$$u_1 = \frac{1}{4}(1 + 2.88 + 0 + 5.59) = 2.37$$

$$u_2 = \frac{1}{4}(4 + 6.13 + 2.37 + 9.87) = 5.59$$

$$u_3 = \frac{1}{4}(9 + 9.88 + 5.59 + 15) = 9.87$$

$$u_4 = \frac{1}{4}(2.37 + 3.01 + 0 + 6.13) = 2.88$$

$$u_5 = \frac{1}{4}(5.59 + 6.16 + 2.88 + 9.88) = 6.13$$

$$u_6 = \frac{1}{4}(9.87 + 9.51 + 6.13 + 14) = 9.88$$

$$u_7 = \frac{1}{4}(2.88 + 3 + 0 + 6.16) = 3.01$$

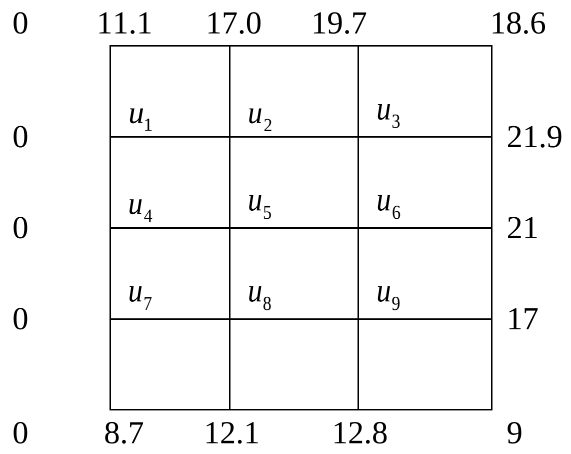
$$u_8 = \frac{1}{4}(6.13 + 6 + 3.01 + 9.51) = 6.16$$

$$u_9 = \frac{1}{4}(9.88 + 9 + 6.16 + 13) = 9.51$$

Step 3: Since in the third and fourth iterations all the values of $u_{i,j}$ at the grid points are same the iteration process is stopped.

Hence
$$\begin{matrix} u_1 = 2.37 & u_2 = 5.59 & u_3 = 9.87 \\ u_4 = 2.88 & u_5 = 6.13 & u_6 = 9.88 \\ u_7 = 3.01 & u_8 = 6.16 & u_9 = 9.51 \end{matrix}$$

Problem 2. Solve the Laplace equation $\nabla^2 u = 0$ at the interior points of the square region given in Figure



Figure

Solution. Let $u_1, u_2, \ldots, u_9$ be the interior grid points and we use Leibmann's iteration process to find these values.

$$u_5 = \frac{1}{4}(17.0+12.1+0+21.0) = 12.5 \quad \text{(SFPF)}$$

$$u_1 = \frac{1}{4}(17.0+0+0+12.5) = 7.4 \quad \text{(DFPF)}$$

$$u_3 = \frac{1}{4}(18.6+12.5+17.0+21) = 17.3 \quad \text{(DFPF)}$$

$$u_7 = \frac{1}{4}(12.5+0+0+12.1) = 6.2 \quad \text{(DFPF)}$$

$$u_9 = \frac{1}{4}(21.0+12.1+12.5+9.0) = 13.7 \quad \text{(DFPF)}$$

We use SFPF to find the starting values at the remaining grid points.

$$u_2 = \frac{1}{4}(17.0+12.5+7.4+17.3) = 13.6$$

$$u_4 = \frac{1}{4}(7.4+6.2+0+12.5) = 6.5$$

$$u_6 = \frac{1}{4}(17.3+13.7+12.5+21) = 16.1$$

$$u_8 = \frac{1}{4}(12.5+12.1+6.2+13.7) = 11.1$$

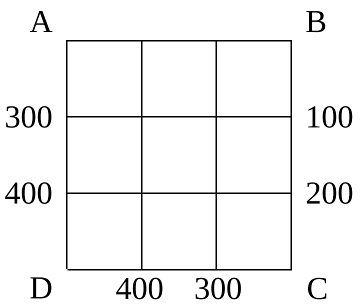
The rought values of the grid points are improved by Liebmann's the iteration formula and the iterations are shown in the following table using standard five point formula.

$\therefore$ Thus $u_1 = 7.9$ $\quad u_2 = 13.7$ $\quad u_3 = 17.9$

$u_4 = 6.6$ $\quad u_5 = 11.95$ $\quad u_6 = 16.3$

$u_7 = 6.7$ $\quad u_8 = 11.2$ $\quad u_9 = 14.3$

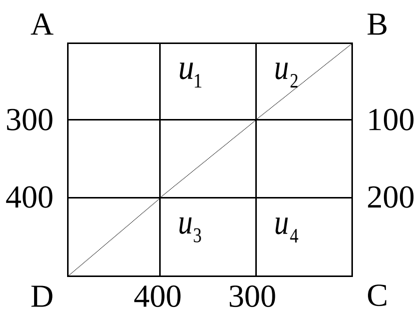
Problem 4. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown below using Liebmann method or otherwise.



Figure

Solution: Let the interior grid points be $u_{1,}u_{2,}u_{3,}u_4$

From the boundary values we notice that they are symmetrical about the diagonal BD. Hence the values at the two boundary points AB are 200 and 100.



Figure

Here $u_1 = u_4$. Hence we need to find u_1, u_2 and u_3 only. Since the corner values are not known we cannot use DFPF. Hence we first assume a suitable approximate value for u_1 and get the other rough values for u_2 and u_3 by using SFPF.

Assume an initial rough value $u_1 = 100 + \frac{2}{3}(300 - 100) = 233$

Rough values:

$u_1 = u_4 = 233$

$u_2 = \frac{1}{4}(100 + u_4 + u_1 + 100) = 167 \qquad (\because u_1 = u_4 = 233)$

$u_3 = \frac{1}{4}(u_1 + 400 + 400 + u_4) = 317$

The rough values are entered in the table and the Liebmann's interations are shown in the table.

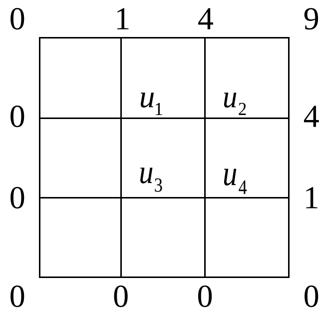
	200	100	
300	u_1 233 246 248 250 250 250	u_2 167 170 174 175 175 175	100
400	u_3 317 320 324 325 325 325	u_4 233 248 250 250 250 250	200
	400	300	

Figure

Hence $u_1 = 250 = u_4$; $u_2 = 175$; $u_3 = 325$

Problem 5. Applying Liemann's method approximate the solution of Laplace's equation $\nabla^2 u = 0$ with $h = \frac{1}{3}$ in a square with the vertices *A*(0,0), *B*(0,1), *C*(1,1), *D*(1,0) given that the unknown function on the boundary is $u(x, y) = 9x^2y^2$

Solution: At the boundary points of the square mesh $h = \frac{1}{3}$ the boundary values are given is the figure.



Figure

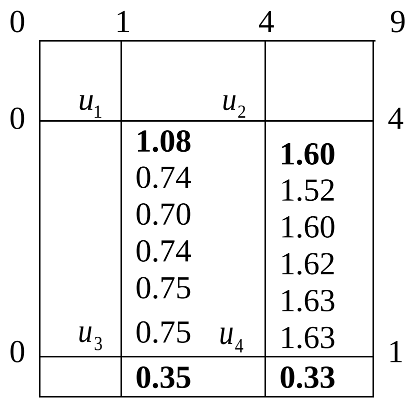
We observe that there is no mid-point in the grid. Hence to initialise the calculations, we assume an initial value for u_4

Let $u_4 = 0.33$. Then $u_1 = \frac{1}{4}(4+0+0+0.33) = 1.08$ (DFPF)

$$u_2 = \frac{1}{4}(4+0.33+1.08+1) = 1.60 \quad \text{(SFPF)}$$

$$u_3 = \frac{1}{4}(1.08+0+0+0.33) = 0.35 \quad \text{(SFPF)}$$

The rough values are entered in the table in bold letters and the Liebmann iterations are shown in the table.



	0.27	0.70
	0.35	0.74
	0.37	0.75
	0.38	0.75
	0.38	0.75

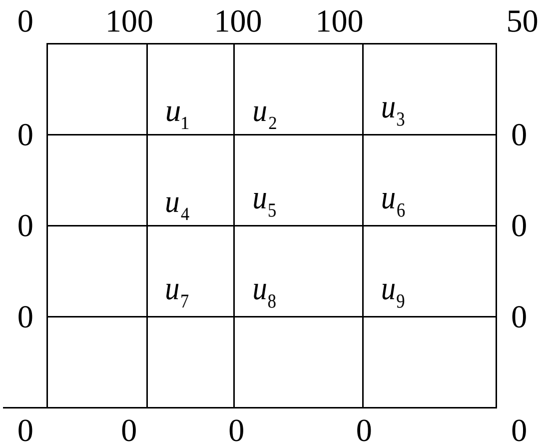
0 0 0 0

Figure

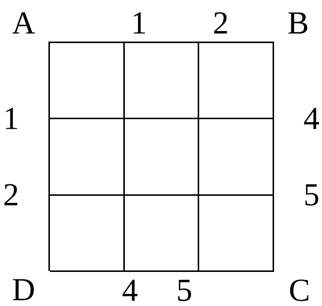
Hence $u_1 = 0.75$; $u_2 = 1.63$; $u_4 = 0.75$; $u_3 = 0.38$.

Exercises

1. Solve the Laplace equation $\nabla^2 u = 0$ at the interior points of the square region given in Figure



2. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown below using Liebmann method or otherwise.



5.4 Poisson's equation

An equation of the form

$$u_{xx} + u_{yy} = f(x,y) \qquad \ldots\ldots. (1)$$

is called Poisson's equation.

The standard five-point formula for (1) takes the form

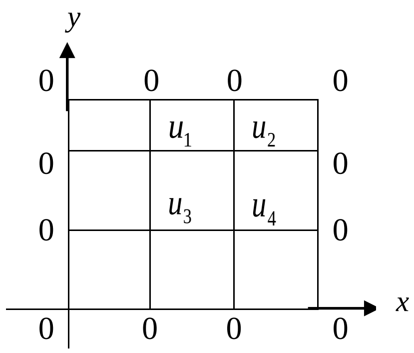
$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) \qquad \ldots\ldots (2)$$

By applying (2) at each interior mesh point we obtain a system of linear equations which can be solved by Gauss Seidel method.

Solved Problems

Problem 1: Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x = 0 = y$; $x = 3 = y$; with $u = 0$ on the boundaries and mesh length 1.

Solution: Here $h = 1$. The region of solution of the given partial differential equation with the boundary values are given in the table.



Figure

Let u_1, u_2, u_3, u_4 be the values at the interior grid points. The standard five point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \qquad \ldots (1)$$

Using (1) at $u_i(i = 1, j = 2)$ we have

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = -10(1^2 + 2^2 + 10)$$

$$0 + u_2 + u_3 + 0 - 4u_1 = -150$$

(i.e) $u_2 + u_3 + 0 - 4u_1 = -150$ … (2)

Using (1) at $u_2 (i = 2, j = 2)$ we have

$$u_1 + u_4 - 4u_2 = -180 \quad \dots (3)$$

Using (1) at $u_3 (i = 1, j = 1)$ we have

$$u_1 + u_4 - 4u_3 = -120 \quad \dots (4)$$

Using (1) at $u_4 (i = 2, j = 1)$ we have

$$u_2 + u_3 - 4u_4 = -150 \quad \dots (5)$$

Solving (2), (3), (4) and (5) we get the values of u's at the four grid points. We use Gauss-Seidel method to solve them.

From (2) and (5) we get $u_1 = u_4$

Hence (2), (3), (4) and (5) can be reduced to

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \quad \dots (6)$$

$$u_2 = \frac{1}{4}(2u_1 + 180) = \frac{1}{2}(u_1 + 90) \quad \dots (7)$$

$$u_3 = \frac{1}{2}(u_1 + 60) \quad \dots (8)$$

Starting with the first iteration using the first approximations $u_2 = 0$ and $u_3 = 0$ we get the following tabular values.

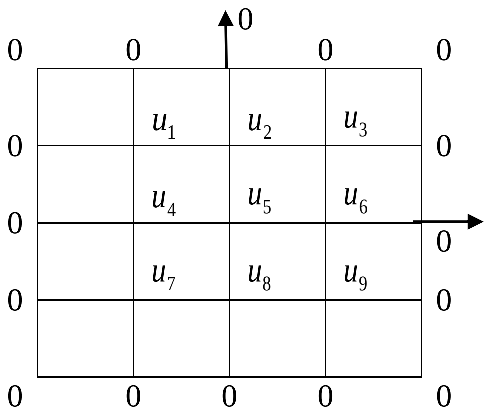
Iterations	1	2	3	4	5	6	7	8	9
$u_4 = u_1$	-	37.5	65.63	72.66	74.42	74.86	74.97	75.0	75
u_2	0	63.75	77.82	81.33	82.21	82.43	82.49	82.50	82.5
u_3	0	48.75	62.82	66.33	67.21	67.43	67.49	67.50	67.5

Since the values in the 8th and 9th iterations are equal the iteration process stops and the values are

$u_1 = 75 = u_4$; $u_2 = 82.5$; $u_3 = 67.5$

Problem 2: Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ in the square mesh given u = 0 on the four boundaries dividing the square into 16 subsquares of length 1 unit.

Solution: Here h = 1. The region of solution of the given Laplace equation with the boundary values are given in the table.



Figure

Let $u_1, u_2, u_3, u_4 .....u_9$ be the values of u at the interior grid points.

Choose coordinate system with origin at the centre u_5 of the square mesh.

We note that the given Poisson partial differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ is symmetrical about x and y axes and also about the line $y = x$.

Hence we have $u_1 = u_3 = u_7 = u_9$ and

$$u_2 = u_4 = u_6 = u_8$$

Hence we have to find u_1, u_2, u_5 only.

The standard five point formula for the given Poission equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 8i^2 j^2 \qquad ..... (1)$$

Using (1) at $u_7(i = -1, j = -1)$ we have

$$u_{-2,-1} + u_{0,-1} + u_{-1,-2} + u_{-1,0} - 4u_{-1,-1} = 8(-1)^2(-1)^2$$

(i,e.) $0 + u_8 + 0 + u_4 - 4u_7 = 8$

(i.e) $u_2 + u_2 - 4u_1 = 8$

(i.e) $u_2 - 2u_1 = 4$ ….. (2)

Using (1) at $u_2(i=0, j=1)$ we have

$$u_{-1,1} + u_{1,1} + u_{0,0} + u_{0,2} - 4u_{0,1} = 8\times 0\times 1 = 0$$

(i.e). $u_1 + u_3 + u_5 + 0 - 4u_2 = 0$

$$\therefore 2u_1 - 4u_2 + u_5 = 0 \quad \text{…… (3)}$$

Using (1) at $u_5(i=0, j=0)$ we have

$$u_{-1,0} + u_{1,0} + u_{0,-1} + u_{0,1} - 4u_{0,0} = 0$$

$$u_4 + u_6 + u_8 + u_2 - 4u_5 = 0$$

$$4u_2 - 4u_5 = 0.$$

(i.e) $u_2 = u_5$ ….. (4)

Solving these equations (2), (3), (4) we get $u_1 = -3$; $u_2 = -2$; $u_5 = -2$

$\therefore$ The solution to the given Poission equation at the 9 interior mesh points.

$$u_1 = u_3 = u_7 = u_9 = -3$$

$$u_2 = u_4 = u_6 = u_8 = -2 \text{ and } u_5 = -2$$

Exercises

1. Solve the Poisson equation $u_{xx} + u_{yy} = -x^2 y^2$ over the square region bounded by the lines x = 0, y = 3 given that u = 10 throughout the boundaries taking h = 1
2. Solve $\nabla^2 u = x^2 + y^2$ in the square region given that u = 2 on the four boundaries dividing the square into 16 subsquares of length one unit.